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Pattern Formation and Nonlinear Dynamics

Set 3. One-dimensional maps

The problems will be discussed in the seminar on Monday, 3 May 2021.

1. Topological conjugation.

We consider the tent map and its conjugations by two maps $k_1(x) = x^2$ und $k_2(x) = \log(2-x)/\log 2$. Which dynamics are obtained by conjugation with respect to $k_1(x)$, $k_1^{-1}(x)$, $k_2(x)$, and $k_2^{-1}(x)$?

2. Variations of the Logistic Map.

Consider the map

$$L(x) = K - a\,x^2 \qquad \text{with } a, K \in \mathbb{R}\,.$$

a) How can its dynamics be mapped to the one of the logistic map $\mu x(1-x)$?

b) Write a Sage- or Python-Program that plots the bifurcation diagram for the logistic map as follows: Incrementing μ in small steps from 0 to 5, take five different initial conditions at a time, iterate them for 400 times, and plot the points (μ, x) for the next 90 iterates.

c) Determine the corresponding bifurcation diagram for $L(x)$ as function of K for fixed a, and function of a for fixed K. How are the three plots related?

3. Deterministic Diffusion.

In the lecture we constructed a 1D maps that describes diffusion on $\mathbb{R}$ with a hopping probability of 1/3 between neighboring intervals that are bounded by integers.

a) Determine a map where the hopping probabilities are 1/2.

b) Determine a map where the probability for a jump to the right is twice as large as the one for going left.

c) Write a Sage- or Python program that plots the maps for $-5 < x < 6$, and follows the evolution of a trajectory that start at $x_0 \in [0,1]$ by plotting

$$(x_0, 0)\,,\ (x_0, x_1)\,,\ (x_1, x_2)\,,\ (x_2, x_2)\,,\ (x_2, x_3)\,,\ (x_3, x_3)\,, \cdots, (x_{20}, x_{21})$$

Add three more trajectories that start at a different initial condition in $[0,1]$.

d) Make histogram based on 1000 trajectories that start in $[0,1]$ and provides the number n_i of trajectories that reside in $[i, i+1]$ after a give number of iterations.
Be prepared to share your screen so that we can vary the parameters on the spot.
How do the histograms differ for the maps constructed in a) and b)?

4. Symbolic Dynamics.

We consider the map $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} (\eta - 1) + \eta\, x & \text{for} \quad x < 0 \\ (1 - \eta) + \eta\, x & \text{for} \quad x \geq 0 \end{cases}$$

a) Determine the set of points that do not leave the interval $[-1, 1]$ in N iterations. How does it look like for $\eta > 2$? How for $1 < \eta < 2$.

b) Show that x can be expressed as

$$x = (\eta - 1) \sum_{k=1}^{\infty} \sigma_{k-1}\, \eta^{-k} \qquad \text{with} \qquad \sigma_k = \operatorname{sign}(f^k(x))$$

c) Show that the trajectories that share the same ν symbols $\{\sigma_k, k = 0 \ldots \nu - 1\}$ lie in an interval of length $\eta^{-\nu}$. What does this imply about the convergence of a symbol sequence towards the associated value x?

d) Determine the total length of the intervals that contain all distinct initial symbol sequences of length ν. How does the total length scale with ν? What does this imply about the uniqueness of the decomposition?

e) Find an example of a point $x_* \in (-1, 1)$ that for a given fixed η has at least two different representations in the form indicated in (b).

f) Determine the symbols σ_k based on the iteration $f^k(x)$. What do you find? What does this imply about the symbol sequences adopted by the symbolic dynamics?