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Pattern Formation and Nonlinear Dynamics

Set 2. Basics in Dynamical Systems

The problems will be discussed in the seminar on Monday, 26 April 2021.

1. Dimensional analysis and equations of motion

Consider a ball of mass m that can move without friction along a tube of mass M and length L. The tube is free to swing around a pivot at one of its end. Its angle with respect to the horizontal axis will be denoted as θ, and the position of the ball inside the tube as x, with origin at the pivot. We consider the initial condition $\theta_0 = 0$, $x_0/L = 0$, and zero intial velocities.

a) Let η be the fraction of the tube that the ball has traversed by the time the tube becomes vertical. Does η depend on L?
Hint: This problem should be solved by dimensional analysis.

b) Determine the equations of motion of $\theta(t)$ and $x(t)$ by the Lagrange formalism. After choosing appropriate length and time units, the solution depends on a single dimensionless parameter. What does this say about the value of η?

c) Employ Sage to numerically solve the equations of motion. How do the trajectories look like? How does η depends of the dimensionless parameter?

2. Phase-space flows and Poincaré maps for linear dynamical systems

We consider the linear dynamical systems

$$\ddot{x}(t) + a\,\dot{x}(t) + b\,x(t) + c \qquad \text{with} \quad a, b, c \in \mathbb{R}\,.$$

and explore their solutions for different choices of a, b, and c?

a) For our analysis the parameter c may always be taken to be zero!
Why is this justified?

b) Consider the case $b = c = 0$. What type of motion is this?
How do the solutions $x(t)$, $\dot{x}(t)$ look like?
Sketch the solutions in the phase space $x, \dot{x}$).

c) Consider the case $a = c = 0$ and $b > 0$. What type of motion is this?
How do the solutions $x(t)$, $\dot{x}(t)$ look like?
Sketch the solutions in the phase space $(x, \dot{x})$.

d) Consider the case $a = c = 0$ and $b < 0$. What type of motion is this?
How do the solutions $x(t)$, $\dot{x}(t)$ look like?
Sketch the solutions in the phase space $(x, \dot{x})$.

e) Consider the case $c = 0$ and $b^2 > 4\,a$. What type of motion is this?
How do the solutions $x(t)$, $\dot{x}(t)$ look like?
Sketch the solutions in the phase space $(x, \dot{x})$.

f) Consider the case $c = 0$ and $b^2 = 4\,a$. What type of motion is this?
How do the solutions $x(t)$, $\dot{x}(t)$ look like?
Sketch the solutions in the phase space $(x, \dot{x})$.

g) Consider the case $c = 0$ and $b^2 > 4\,a$. What type of motion is this?
How do the solutions $x(t)$, $\dot{x}(t)$ look like?
Sketch the solutions in the phase space $(x, \dot{x})$.

h) In case (g) almost all trajectories intersect the positive x-axis infinitely often. In such a situation the dynamics can be characterized by the Poincaré map $f(x)$ that provides the value of the $(n+1)^{\text{st}}$ intersection as function of the n^{th} intersection, i.e., $x_{n+1} = f(x_n)$.
How does the Poincaré map look like in this case?

3. Van der Pol oscillator

In this exercise we work out and complement the treatment of the van der Pol oscillator provided in Takashi Kanamaru (2007), Scholarpedia, 2(1):2202. For a tunnel diode with $I = \phi(V) = \gamma\, V^3 - \alpha\, V$ he provides the equations (cf. the section Electrical Circuit)

$$C\,\dot{V} = -\phi(V) - W \tag{1a}$$

$$L\,\dot{W} = V \tag{1b}$$

a) Determine the equation of motion of $LC\,\ddot{V}$ in terms of $\dot{V}$ and V.
Hint: Multiply (1a) by L, take the time derivative, and eliminate W by means of (1b).
Absorb $\sqrt{LC}$ into the time scale and show that for every fixed reference voltage V_0 the dimensionless variable $x = V/V_0$ evolves according to

$$0 = \ddot{x} + a\,V_0^2\,\left(x^2 - \frac{b}{V_0^2}\right)\,\dot{x} + x$$

where the constants a and b depend on L, C, α, and γ.
Determine a and b.

b) Kanamaru discusses the dimensionless van der Pol equation

$$0 = \ddot{x} - \epsilon\,(1 - x^2)\,\dot{x} + x$$

How does he choose ϵ and V_0?
What does he assume about the parameters L, C, α, and γ to arrive at this equation?
When the assumption does not hold, one has to add a minus sign to the equation. Where is it needed?

c) For the bifurcation analysis I rather adopted

$$0 = \ddot{x} - (\eta - x^2)\,\dot{x} + x \tag{2}$$

Which choice of η and V_0 is needed to arrive at this equation?
I claim that one can *always* write the evolution in this form. However, under some conditions one has to change the definition of the dimensionless time. How? What are the physical consequences of this change?

d) The phase-space solutions of (2) can more tranparently be discussed by rewriting the second order ODE for x in terms of two coupled first-order equations. To this end we consider the auxiliary variable $y = x\,(\eta - x^2/3) - \dot{x}$. Evaluate the time derivative of y to verify that $\dot{y} = x$.
Show that this provides

$$\dot{x} = x\,\left(\eta - \frac{x^2}{3}\right) - y$$
$$\dot{y} = x$$

Hint: Take the time derivative of the first equation and eliminate $\dot{y}$ by use of the second, and check out the position of the fixed point to verify that there is not offset of x.

e) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{R^2}{2} = x^2\,\left(\eta - \frac{x^2}{3}\right)$$

$$\text{and}\quad \dot{\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\arctan\frac{y}{x} = 1 - \frac{1}{2}\,\sin(2\theta)\,\left(\eta - \frac{x^2}{3}\right)$$

Hint: The equation for the angle can be derived with moderate effort by working out the time derivative of $\tan\theta$. Discuss the time evolution for small η. Argue that $x^2/3\eta$ will then be of order unity close to the limit cycle, such that R changes slowly and θ is almost constant. How does the flow look like in the (x, y) plane.

f) Sketch the nullclines for $\dot{x} = 0$ and $\dot{y} = 0$ in the (x, y) phase space. Where would you place the limit cycle for small positive η?
How does the flow look like for large η?
What happens when η changes sign?

g) Verify your results by numerical plots of the phase-space flow.
Use your simulation to also plot the Poincaré map of subsequent crossings of the positive x axis.