Homework Exercises 10

A PDF-file of your solution to the problems 9.1 – 9.3 should uploaded to your Moodle account

by Sunday, January 31 (with a grace time till Monday at noon).

The parts marked by ⋆⋆ are suggestions for further exploration that will be followed up in the seminars. They should not be handed in with your solution.

Problems

Problem 10.1. Evolution of a particle in a Mexican-hat potential

We explore the motion of a particle of mass m in a rotation-symmetric potential

$$\Phi(r) = \frac{m\,A}{4}\, r^2\, (r^2 - 2\, r_0^2)$$

The particle evolves in a plane where its position is specified by the polar coordinates (r, θ).

a) Sketch the potential. Where are its maxima and minima?

b) Determine the Lagrange function for this problem, and determine the equations of motion for $\theta(t)$ and $r(t)$.
Bonus. The angular momentum and the energy of the particle are conserved. How do you see this without calculation based on the Lagrange function?

c) Determine a frequency ω, a length scale ℓ and a constant K, such that

$$\frac{\mathrm{d}^2\hat{r}}{\mathrm{d}(\omega t)^2} = \hat{r} - \hat{r}^3 + \frac{K}{\hat{r}^3}$$

where $\hat{r}$ denotes the dimensionless (scalar) distance

$$\text{with } \hat{r}(t) = \frac{r(t)}{\ell}\,.$$

In the following we discuss the dimensionless equations, where we absorb ω into the time scale and drop the hat to avoid clutter in the equations.

d) Multiply the equation of motion by $\dot{r}$, and rewrite it in the form

$$E = \frac{\dot{r}^2}{2\omega^2} + V_{\text{eff}}(r) \qquad \text{with} \qquad V_{\text{eff}}(r) = \frac{r^4}{4} - \frac{r^2}{2} + \frac{K}{2\,r^2}\,.$$

e) Sketch the effective potential $V_{\text{eff}}(r)$ and the phase portrait of the motion for $K > 0$.

Bonus. Why is it necessary to give a separate discussion of $K = 0$?

Problem 10.2. Foucault Pendulum

A pendulum of mass M that is suspended at a chord of length ℓ in a constant gravity field with acceleration $-g\hat{z}$. We choose Cartesian coordinates for the description of its motion such that the pendulum is at rest in the origin of the coordinate system. The mass of the chord is negligible as compared to its mass such that ℓ is the distance between the pendulum fulcrum and its center of mass.

a) Assume that the pendulum moves in an inertial frame, and that it is performing oscillations with a small amplitude A around its rest position. Sketch the setup.

b) We describe the motion of the pendulum now by only following its x and y coordinate. Determine the kinetic energy T and potential energy V of the pendulum as function of x and y.

c) Perform a Taylor expansion of T and V for small x/ℓ and y/ℓ to show that

$$T \simeq \frac{M}{2}\,\left(\dot{x}^2 + \dot{y}^2\right)\,, \qquad\qquad V \simeq \frac{Mg}{2\ell}\,\left(x^2 + y^2\right)\,.$$

What are the leading order corrections to this equation? Under which condition is it admissible to disregard this correction?

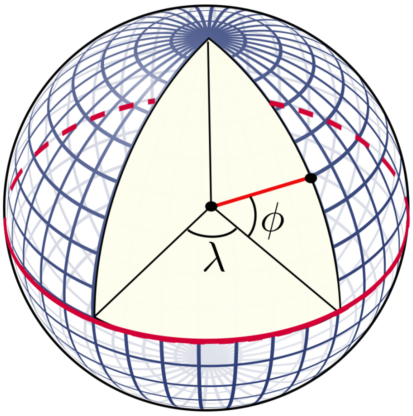
d) Show that the equations

$$\ddot{x} = -\frac{g}{\ell}\,x\,, \qquad\qquad \ddot{y} = -\frac{g}{\ell}\,y\,.$$

provide a faithful description for the motion of the pendulum.
Sketch the solutions in phase space.
Provide the solution for pendulum that is released with zero velocity from a position (x_0, y_0).

⋆ e) What is the qualitative difference between the solutions here, and those obtained for the spherical pendulum?

f) We explore now how the motion of the pendulum changes when one accounts for the motion of Earth that is spinning with frequency Ω. To this end we take into account the additional forces acting on the pendulum when it is set up at the latitude ϕ (see sketch to the left). Let the directions $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be oriented towards East, North, and radially outwards from the Earth center. Determine the explicit form of these vectors in terms of the anges λ and ϕ defined in the sketch.



[Peter Mercator, Public Domain, wikimedia]

g) Due to the Earth rotation the coordinate vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are explicitly time dependent. Verify that this time dependence can be expressed in terms of a cross procduct

$$\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} = \mathbf{\Omega} \times \hat{\mathbf{x}} \qquad \frac{\mathrm{d}\hat{\mathbf{y}}}{\mathrm{d}t} = \mathbf{\Omega} \times \hat{\mathbf{y}} \qquad \frac{\mathrm{d}\hat{\mathbf{z}}}{\mathrm{d}t} = \mathbf{\Omega} \times \hat{\mathbf{z}}$$

where $\mathbf{\Omega}$ is a vector oriented from the Earth center to the North pole whose absolute value amounts the Earth rotation frequency. Consequently, we also have $\lambda = |\mathbf{\Omega}|\,(t - t_0)$, and ϕ is constant for a pendulum at a fixed position on the Earth surface.

h) We take the Earth cente of the origin of the coordinate system, and write the position of the pendulum weight as

$$\mathbf{q}(t) = R\,\hat{\mathbf{z}}(\Omega t) + x(t)\,\hat{\mathbf{x}}(\Omega t) + y(t)\,\hat{\mathbf{y}}(\Omega t)$$

Vertical motion will again be beglected such that R is constant. The frequency of the pendulum is much faster than Ω. Therefore, we also neglect terms of order Ω^2. Show that the equation derived in d) picks then up additional terms as follows

$$\ddot{x} = -\frac{g}{\ell}\,x + \Omega\,\dot{y}\,\sin\phi$$

$$\ddot{y} = -\frac{g}{\ell}\,y - \Omega\,\dot{x}\,\sin\phi$$

This linear ODE can be solved along the lines that we discussed in previous lectures. However, there is a special symmetry in the equation that can be exploited to find the solution in a much more elegant way:

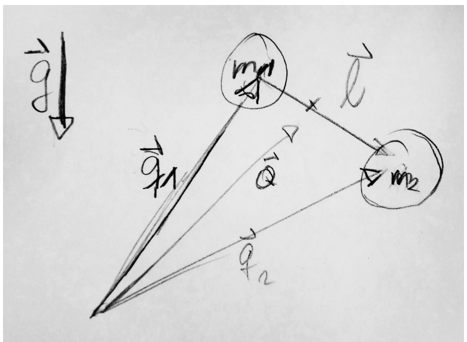
i) Consider the complex variable $\mathsf{z} = x + \mathrm{i}y$,[1] and demonstrate that its equation of motion is a complex homogeneous linear ODE

$$\ddot{\mathsf{z}} + \frac{g}{\ell}\,\mathsf{z} + \Omega\,\dot{\mathsf{z}}\,\sin\phi = 0$$

j) Consider the Ansatz $\mathsf{z}(t) = A\,\exp(pt)$ with constant complex numbers p and A to solve the equation of motion. There will be two choices $p_\pm$ that solve the equation.

$\star$ k) Determine the solution for an initial condition where the pendulum is released at rest from the position $(x_0, 0)$.
Demonstrate that the solution describes a pendulum that is swinging in a plane which is slowly rotating around the vertical axis. Determine the rotation frequency.

Problem 10.3. Flight of a dumbbell



We explore the flight of a dumbbell under the influence of gravity $\mathbf{g}$ in our three-dimensional space. The dumbbell is idealized as two particles of masses m_1 and m_2. They positions $\mathbf{q_1}(t)$ and $\mathbf{q}_2(t)$ will be kept at a fixed distance ℓ by a bar of negliglible mass. We denote the center of mass of the dumbbell as $\mathbf{Q}$ and the relative coordinate as $\boldsymbol{\ell} = \mathbf{q_2} - \mathbf{q}_1$.

a) We express the relation between $(\mathbf{Q}, \boldsymbol{\ell})$ and the positions $\mathbf{q}_i$, $i \in \{1, 2\}$ as $\mathbf{q}_i = \mathbf{Q} + \alpha_i\,\boldsymbol{\ell}$. Determine the real numbers α_i, $i \in \{1, 2\}$.

b) Show that the kinetic energy and the potential energy of the dumbbell have the form

[1]Beware the font: The complex variable z must not be confused with the vertical coordinate z!

$$T = \frac{M}{2}\,\dot{\mathbf{Q}}^2 + \frac{\mu}{2}\,\dot{\boldsymbol{\ell}}^{\,2}\,,$$
$$V = -M\mathbf{g}\cdot\mathbf{Q} + \Phi(\ell)$$

where $\Phi(\ell)$ is a potential that will generate the force fixing the distance of the masses to the value ℓ. How do M and μ depend on m_1 and m_2?

c) Show that

$$\ddot{\mathbf{Q}} = \mathbf{g}$$

How does the trajectory of the center of mass of the dumbbell look like when the dumbbell is thrown at time t_0 from a position $\mathbf{Q}_0$ with a velocity $\mathbf{V}_0$?

d) Show that

$$\mu\,\ddot{\boldsymbol{\ell}} = -\hat{\boldsymbol{\ell}}\cdot\nabla\Phi(\ell) \quad \text{with} \quad \hat{\boldsymbol{\ell}} = \frac{\boldsymbol{\ell}}{\ell}\,.$$

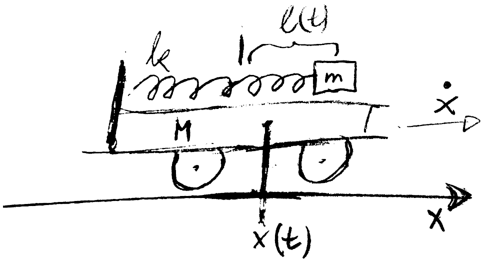
e) Show that the energy $E = \mu\dot{\boldsymbol{\ell}}^2/2 + \Phi(\ell)$ and the angular momentum $\mathbf{L} = \mu\boldsymbol{\ell}\times\dot{\boldsymbol{\ell}}$ are constants of the motion of the dumbbell.

f) Write the rotational motion of the dumbbell as $\dot{\boldsymbol{\ell}} = \boldsymbol{\Omega}\times\boldsymbol{\ell}$ with a constant vector $\boldsymbol{\Omega}$. Verify by explicit calculations that this ansatz fulfills the requirements on the conservation of the distance between the masses, the energy and the angular momentum.

g) Provide the position of the masses $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$ for the initial conditions provided in (c), some fixed $\boldsymbol{\Omega}$, and $\boldsymbol{\ell}_0$.

Self Test

Problem 10.4. Spring on rails



We consider a cart that moves without friction on a horizontal track. It has mass M, and at time t it is located at position $x(t)$. On the cart we attach a weight of mass m to a spring with spring constant k. It oscillates without friction in the track direction, and its displacement from the rest position is denoted as $\ell(t)$.

a) Determine the kinetic energy of the cart, the kinetic energy of the weight, and the potential energy due to the tension of the spring.
Provide the resulting Lagrange function for the oscillator on the cart.

b) Identify the equation of motion for x, and show that it leads to a conserved quantity of the form

$$P = \alpha\,\dot{x} + \beta\,\dot{\ell}$$

How do α and β depend on the parameters m, M, and k?

c) Determine the x-component Q of the center of the system. Which interpretation does this provide for the result of part (c)?

* d) In the following we work in the center of mass frame. Show by and explicit calculation that the Lagrangian can then be written as

$$\mathcal{L} = \frac{a}{2}\dot{Q}^2 + \frac{b}{2}\dot{\ell}^2 + \frac{c}{2}\ell^2$$

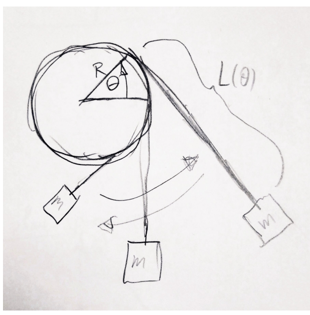
How do a, b and c depend on the parameters m, M, and k?

* e) Can you provide the values of a, b, and c without calculation?

f) Determine and solve the equations of motion, and provide the explicit expressions $Q(t)$, $x(t)$, and $\ell(t)$ for the following setting: initially the spring is stretched to a value ℓ_0, and the system does not move. How will it evolve when it is released at time t_0?

* g) Discuss the frequency of the oscillations for the limiting cases $m \gg M$ and $m \ll M$: Which physical argument provides these frequencies without calculation?

Problem 10.5. The cylinder pendulum



We consider a pendulum that is built by attaching a chord to a cylinder (e.g. a broom stick), wrapping it a few times around the cylinder, and attaching a mass m to the loose end of the chord. When the mass oscillates, the length of the chord is changing because it is wrapped around the cylinder. We describe the position where the chord touches the cylinder by the angle θ. For $\theta = 0$ the chord is entirely wrapped around the cylinder, and the mass will be located at the position $(R, 0)$.

a) Let $L(\theta)$ be the length of the chord from the cylinder to the mass, as indicated in the figure. Argue that $L(\theta)$ is proportional to θ! What is the proportionality constant? What are admissible values for θ?

b) Employing polar coordinates based on the angle θ, where the chord touches the cylinder, we write the position of the mass as

$$\mathbf{q}(t) = q_R\, \hat{\mathbf{R}}(\theta) + q_\theta\, \hat{\boldsymbol{\theta}}(\theta)$$

How do q_R and q_R depend on θ and on the parameters of the problem when we assume that the chord is stretched between the cylinder and the mass (cf. (h)!).

c) Show that

$$\dot{\mathbf{q}} = c\, \theta\, \dot{\theta}\, \hat{\mathbf{R}}$$

with some real constant c. How is c related to the parameters?

d) Determine the kinetic energy, the potential energy, and the Lagrange function.

e) Determine the equation of motion for θ.

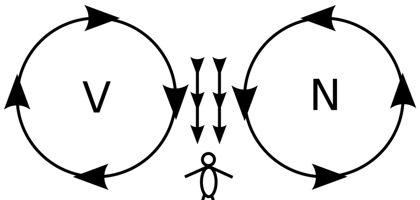
f) What are the fixed points of the dynamics? Provide a physical argument for their stability (no calculation!).

g) What are the values of the potential energy right at the unstable fixed points? Use (only!) this information to sketch the phase-space trajectories starting from unstable fixed points in the interval $0 < \theta < 7\pi$. Complete the phase-space portrait of the dynamics by adding trajectories that reside in the vicinity of the stable fixed points.

*h) Mark the region in the phase space where the assumption of (b), that the chord is stretched, is not justified.

Bonus Problem

Problem 10.6. Wind Systems on Earth



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On length scales beyond 100 km the mean horizontal wind on Earth is significantly influenced by the Coriolis force. One consequence is expressed by Buys Ballot's law, which was taught to Naval Cadets as:

"In the Northern Hemisphere, if you turn your back to the wind, the low pressure center will be to your left and somewhat toward the front." (Aerology for Pilots, McGraw-Hill, 1943, pg 43)
In the following we explore how the Coriolis force shapes the main wind systems on Earth, and how the Buys Ballot's law comes about.

a) At the equator warm air rises and moves to towards the poles at high altitudes, while cool air moves towards the equator along the ground. We say that the rise of air induces a low pressure region that is sucking air towards the equator. The mean flow is deflected by the Coriolis force. In the vicinity of the equator this gives rise to trade winds.
From which direction will the winds coming from the North and from the South approach the equator?

b) The velocity, i.e. the speed and the direction, of the flow will no longer change when the acceleration of the wind by the pressure gradient and by the Coriolis force balance (geostrophic wind). We consider the Euler equation for the momentum-balance of fluid flow to explore this relation

$$\varrho \, \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla P$$

In this equation ϱ is the mass density of air, $\mathbf{u}$ is the flow velocity, and P is the pressure. The equation holds in inertial systems and when the viscosity of the fluid may be neglected. The latter condition holds for the atmosphere. The former condition entails that the time derivative must be augmented by the Coriolis force when adopting a coordinate system that is co-moving with the Earth surface. Demonstrate that condition for a stationary flow velocity $\mathbf{u}$

amounts then to

$$\nabla P = -2\varrho\, \mathbf{\Omega} \times \mathbf{u}$$

Here, $\mathbf{\Omega}$ is the angular velocity of Earth.

c) Determine the pressure difference over a distance of 1000 km when air is moving at a (mean) horizontal speed of 50 km/h. How does the pressure difference depend on latitude? What is the relation between the direction of the pressure gradient and the flow velocity?
Hint: Air has a density of about $\varrho \simeq 1.3\,\mathrm{kg/m^3}$.

d) Consider now the flow along the equator. How does the pressure change upon motion to the West and to the East, respectively? What does this tell about the stability of the wind? Why is this argument incomplete at best and wrong if worst comes to worst?

e) High pressure and low pressure regions at mid latitudes (for instance close to Europe) have typical diameters of 1000 km, and the predominant wind directions are along isobars rather than in the direction of the pressure gradient. Compare the pressure difference determined in c) with typical pressure difference of high pressure and low pressure regions, and discuss the orientation of the flow around the respective regions.