Homework Exercises 9

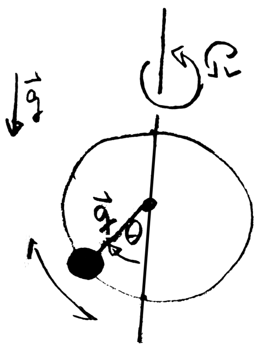
A PDF-file of your solution to the problems 9.1 – 9.3 should uploaded to your Moodle account

by Sunday, January 25 (with a grace time till Monday at noon).

The parts marked by ⋆⋆ are suggestions for further exploration that will be followed up in the seminars. They should not be handed in with your solution.

Problems

Problem 9.1. Phase-space analysis for a pearl on a rotating ring



We consider a pearl of mass m that can freely move on a ring that stands vertically in the gravitational field, and spins with angular velocity Ω around its vertical symmetry axis. Since the position of the pearl is constrained to lie on the surface of a sphere, we adopt spherical coordinates and describes its position as

$$\mathbf{q}(t) = \ell \, \hat{r}\big(\theta(t), \Omega t\big)$$

where θ is the deflection of the pearl from its rest position of the lowermost point of the ring, and $\phi = \Omega t$ the orientation of the ring in the horizontal plane.

a) Determine the components of $\hat{r}(\theta, \phi)$ in Cartesian coordinates where the z-axis is aligned antiparallel to $\mathbf{g}$. Verify then by explicit calculation that $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ with

$$\hat{\theta} = \frac{\partial \hat{r}}{\partial \theta} \qquad \text{and} \qquad \hat{\phi} = \hat{r} \times \hat{\theta}$$

form an orthonormal basis. How is $\hat{\phi}$ related to $\partial \hat{r} / \partial \phi$?

b) Evaluate $\dot{\mathbf{q}}(t) = \ell \, \dot{\hat{r}}(\theta(t), \Omega t)$.

c) Determine the kinetic energy T and the potential energy V of the pearl.

d) Use the Lagrange formalism to determine the equation of motion of the pearl.

e) Determine the fixed points for the motion of the pearl, and discuss their stability as function of Ω.

f) Sketch phase-space plots for the motion of the pearl for small and large values of Ω.

Problem 9.2. A trip on the rollercoaster

We parameterize the height H of a rollercoaster track by the contour length ℓ of the track. In some rough approximation this will lead to a biquadratic function

$$H(\ell) = a\,(\ell^2 - L^2)^2$$

where $H(\ell)$ is the height over the lowermost point of the track.

a) Sketch $H(\ell)/(aL^4)$ as function of ℓ/L.

b) What are the units of a and L? What is the height of the looping? Which distance does a car travel from the lowest point of the track, once through the looping, till it arrives at the following minimum?

Rollercoaster Tornado in the adventure park Hellendoorn, The Netherlands [public domain, wikimedia.commons]

c) We first consider the motion of a single car of mass m on the track. Determine its kinetic and potential energy as function of $\ell(t)$. Verify that its equation of motion takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\ell(t)}{L} = \omega^2\,\frac{\ell(t)}{L}\left(\frac{\ell(t)}{L} - 1\right)\left(\frac{\ell(t)}{L} + 1\right)$$

How is ω related to a, L, and the gravitational acceleration g?

d) Sketch the solutions in phase space.

** e) Consider now two cars of the same mass m. We model them as two point masses that move along the track with a fixed distance $2w$. Let $\ell(t)$ denote the position of the center of mass of the two cars. Show that the kinetic energy and the potential energy will then take the form

$$
\begin{aligned}
T &= m\,\dot{\ell}^2(t) \\
V &= 2\,m\,g\,a\,\left[(\ell^2 - A^2)^2 + B^2\,\ell^2\right]
\end{aligned}
$$

How do A and B depend on w and L?

** f) Discuss also the case of three cars of mass m and distance $2w$.

Problem 9.3. Two masses hanging at a rubber band

Two weights of the same mass m are attached on opposite ends of a rubber band that is hanging over a roll. The weights are at height h_1 and h_2. They move only vertically, either one up and one down at a fixed length of the band, or stretching the band, or releasing tension on the band. We assume that friction and the mass of the band are negligible.

a) Sketch the problem, and indicate the relevant parameters and coordinates.

b) We describe the problem by adopting the coordinates $H = h_1 + h_2$ and $D = h_1 - h_2$. Verify that the Lagrange function will then take the following form

$$
\mathcal{L}(H, D, \dot{H}, \dot{D}) = \frac{\mu}{2}\left(\dot{H}^2 + \dot{D}^2\right) - mg\,H - \frac{k}{2}\,H^2
$$

Here k is the elastic module of the rubber band, and μ is an effective mass. How is μ related to m?

The expression for the $\mathcal{L}$ adopts a particular choice of the origin used to specify h_1 and h_2. Which choice has been used?

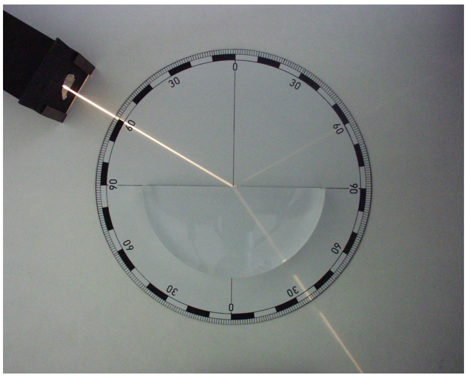
c) Determine the equations of motion for H and D.

d) Solve the equations of motion and interpret the result. For which values of H and D will you trust the result?

Self Test

Problem 9.4. Fermat's principle

Fermat's principle states that a light beam propagates along a path minimizing the flight time. When passing from air into glass it changes direction according to Sellius' refraction law.



© Zátonyi Sándor (ifj.) Fizped (talk) CC BY-SA 3.0, wikimedia commons

Here, we consider a setting where the beam starts in air at the position, $(x, y) = (0, 0)$, to the top left in the figure, with coordinates where $\hat{x}$ points downwards and $\hat{y}$ to the right. The path of the light is described by a function $y(x)$. We require that beam passes from air into the glass at the position (a, u) such that it will eventually proceed through the prescribed position (b, w) in the glass. The speed of light in air and in glass will be denoted as c_L and c_G, respectively.

a) Show that the time of flight T for a (hypothetical) trajectory $y(x)$ with derivative $y'(x)$ can be determined as follows

$$T = c_L^{-1} \int_0^a \mathrm{d}x \, \sqrt{1 + (y'(x))^2} + c_G^{-1} \int_a^b \mathrm{d}x \, \sqrt{1 + (y'(x))^2} \, .$$

b) Determine δT for a variation $y(x) + \varepsilon \, \delta y(x)$ of the trajectory. We describe the glass surface by a function $s(x)$, but we do not know at which position along the surface the beam passes from air into the glass. What does this imply for $\delta y(x)|_{x=0}$, $\delta y(x)|_{x=a}$ and $\delta y(x)|_{x=b}$? What does it imply for the boundary terms that arise from the integration by parts, when determining δT?

c) Show that the beam must go in a straight line in air and in glass. Show that this implies that

$$T(u) = \frac{1}{c_L} \sqrt{u^2 + a^2} + \frac{1}{c_G} \sqrt{(w-u)^2 + (b-a)^2} \, .$$

Derive Snellius' law from the condition that $0 = \mathrm{d}T(u)/\mathrm{d}u$.

Bonus. Snellius' Law can also be directly obtained from Fermat's principle. How?

Problem 9.5. The kitchen pendulum

We consider a pendulum that is built from two straws (length L_1 and L_2), two corks (masses m_1 and m_2), a paper clip, and some Scotch tape (see picture to the right). It is suspended from a shashlik skewer, and its motion is stabilized by means of the spring taken from a discharged ball-pen. Hence, the arms move vertically to the skewer. We denote the angle between the arms as α, and the angle of the right arm with respect to the horizontal as $\theta(t)$.

a) Determine the kinetic energy, T, and the potential energy, V, of the pendulum. Argue that T and V can only depend on θ and $\dot{\theta}$, and determine the resulting Lagrangian $\mathcal{L}(\theta, \dot{\theta})$.

b) Determine the EOM of the pendulum.

c) Find the rest positions of the pendulum, and discuss the motion for small deviations from the rest positions. Sketch the according motion in phase space.

d) The EOM becomes considerably more transparent when one selects the center of mass of the corks as reference point. Show that the center of mass lies directly below the fulcrum when the pendulum it at rest.

e) Let ℓ be the distance of the center of mass from the fulcrum, and $\varphi(t)$ be the deflection of their connecting line from the vertical. Determine the Lagrangian $\mathcal{L}(\varphi, \dot{\varphi})$ and the resulting EOM for $\varphi(t)$. Do you see how the equations for $\ddot{\theta}(t)$ and $\ddot{\varphi}(t)$ are related?

Bonus Problem

Problem 9.6. Stability of soap films

© Mathematikum Gießen

http://mathematikum.df-kunde.de/Wanderausstellung/index.php?m=2&la=de&id=314

When a soap film is suspended between two rings, it takes a cylinder-symmetric shape of minimal surface area. We discuss here the form of the film for rings of radius R_0 and R_1 positioned at the height x_0 and x_1, respectively. At the Mathematikum in Gießen there is a nice demonstration experiment: x_0 is the surface height of soap solution in a vessel around the platform where the children are standing, and x_1 is the height of the ring pulled upwards by the children.

a) Let $w(x)$ be the radius of the cylinder-symmetric soap films at the vertical position x. Sketch the setup and mark the relevant notations for the problem.

b) Show that the surface area A of the soap film takes the form to

$$A = \int_{x_0}^{x_1} \mathrm{d}x \, w(x) \, f(w'(x)) \, ,$$

Here, the factor $f(w'(x))$ takes into account that the area is larger when the derivative $w'(x) = \mathrm{d}x/\mathrm{d}x$ increases. Determine the function $f(w'(x))$ in this expression.

c) Show that A is extremal for shapes $w(x)$ that obey the differential equation

$$w''(x) = \frac{1 + (w'(x))^2}{w(x)} \, .$$

d) Determine the solutions of the differential equation.

Hint: Rewrite the equation into the form

$$\frac{w'(x) \, w''(x)}{1 + (w'(x))^2} = \frac{w'(x)}{w(x)} \, .$$

e) Consider now solutions with $-x_0 = x_1 = a$ and $R_0 = R_1 = R$, and denote the radius at the thinnest point of the soap film as w_0. Show that w_0 is the solution of

$$\frac{R}{a} = \frac{w_0}{a} \cosh \frac{a}{w_0} \,.$$

f) Sketch R/a as function of a/w_0. For given R and a you can then find w_0. For small separation of the rings you should find two solutions. What happens when one slowly rises the ring? Will an adult ever manage to pull up the ring to head height before the film ruptures?