Homework Exercises 8

A PDF-file of your solution to the problems 8.1 – 8.3 should uploaded to your Moodle account

by Sunday, January 17 (with a grace time till Monday at noon).

The parts marked by are suggestions for further exploration that will be followed up in the seminars. They should not be handed in with your solution.

Problems

Problem 8.1. Solving linear 1st order ODEs by decomposition in eigenfunctions

a) Show that every nth order linear ODE for a vector valued function function $\mathbf{x}(t) \in \mathbb{R}^d$ can be written in the form of a homogenous 1st order ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Gamma} = \mathsf{A}\,\mathbf{\Gamma}$$

where $\mathbf{\Gamma} \in \mathbb{R}^{dn}$ and A is an appropriately chosen $dn \times dn$ matrix. The vector $\mathbf{\Gamma}$ will be denoted as phase vector.

b) Consider now the EOM of the harmonic oscillator with damping γ and eigenfrequency ω,

$$\ddot{x} = -\gamma\,\dot{x} - \omega^2\,x$$

In the following $\mathbf{\Gamma} = (x, \dot{x})^T$ will refer to this damped oscillator and A will denote the 2×2 matrix appearing in its EOM.

c) Determine the eigenvalues $\lambda_\pm$ of A.

d) Determine right eigenvectors $\{\mathbf{e}_\pm = (x_\pm, v_\pm)^T\}$ of A, i.e., the vectors

$$\mathsf{A}\begin{pmatrix} x_+ \\ v_+ \end{pmatrix} = \lambda_+ \begin{pmatrix} x_+ \\ v_+ \end{pmatrix} \qquad \text{and} \qquad \mathsf{A}\begin{pmatrix} x_- \\ v_- \end{pmatrix} = \lambda_- \begin{pmatrix} x_- \\ v_- \end{pmatrix}$$

Remark: It is fine when the vectors are not normalized.

e) Determine the left eigenvectors $\{\mathbf{b}_\pm = (\xi_\pm, \nu_\pm)\}$ of A, i.e., the vectors

$$(\xi_+,\, \nu_+)\; \mathsf{A} = \lambda_+\; (\xi_+,\, \nu_+)$$
$$(\xi_-,\, \nu_-)\; \mathsf{A} = \lambda_-\; (\xi_-,\, \nu_-)$$

Remark: It is fine when the vectors are not normalized.

f) Show that

$$\mathbf{b}_+ \cdot \mathbf{e}_- = (\xi_+,\, \nu_+) \cdot \begin{pmatrix} x_- \\ v_- \end{pmatrix} = 0 \qquad \text{and} \qquad \mathbf{b}_- \cdot \mathbf{e}_+ = (\xi_-,\, \nu_-) \cdot \begin{pmatrix} x_+ \\ v_+ \end{pmatrix} = 0$$

g) Verify that every vector $\boldsymbol{\Gamma} \in \mathbb{R}^2$ can be decomposed as

$$\boldsymbol{\Gamma} = g_+\, \mathbf{e}_+ + g_-\, \mathbf{e}_- \qquad \text{with} \quad g_\pm = \frac{\mathbf{b}_\pm \cdot \boldsymbol{\Gamma}}{\mathbf{b}_\pm \cdot \mathbf{e}_\pm}$$

Remark: In g_+ there are only plus signs on the right-hand side, and for g_- only minus signs.

h) The expression derived in g) holds in general. Show that one thus can express the solution of the EOM $\dot{\boldsymbol{\Gamma}}(t) = \mathsf{A}\, \boldsymbol{\Gamma}(t)$ with initial condition $\boldsymbol{\Gamma}_0 = \boldsymbol{\Gamma}(t_0)$ as

$$\boldsymbol{\Gamma}(t) = g_+^{(i)}\; \mathrm{e}^{\lambda_+\,(t-t_0)}\; \mathbf{e}_+ + g_-^{(i)}\; \mathrm{e}^{\lambda_-\,(t-t_0)}\; \mathbf{e}_- \qquad \text{with} \quad g_\pm^{(i)} = \frac{\mathbf{b}_\pm \cdot \boldsymbol{\Gamma}_0}{\mathbf{b}_\pm \cdot \mathbf{e}_\pm}$$

i) How does the solution look like specifically for the damped harmonic oscillator?

Problem 8.2. Taylor series of 2d functions

A potential $\Phi(\mathbf{x})$, is a scalar function that depends on the position $\mathbf{x}$. In general the Taylor expansion of such a scalar function describes the change of the function for small deviations $\boldsymbol{\varepsilon}$ of the position. Specifically, for $\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\varepsilon}$ we have

$$\Phi(\mathbf{x}) = \Phi(\mathbf{x}_0) + (\epsilon_i\, \partial_i)\Phi(\mathbf{x}_0) + \frac{1}{2}\,(\epsilon_i\, \partial_i)\,(\epsilon_j\, \partial_j)\Phi(\mathbf{x}_0) + \frac{1}{3!}\,(\epsilon_i\, \partial_i)\,(\epsilon_j\, \partial_j)\,(\epsilon_k\, \partial_k)\Phi(\mathbf{x}_0) + \dots$$

Here, ϵ_i denotes the i-component of the vector $\boldsymbol{\varepsilon}$ with respect to an orthonormal basis $\hat{\mathbf{e}}_i$, and ∂_i is the partial derivative with respect to the according coordinate x_i of $\mathbf{x}$. Moreover, we use the Einstein convention that requires summation over repeated indices, i.e., $\epsilon_i\, \partial_i$ is an abbreviation for $\epsilon_i\, \partial_i = \sum_i \epsilon_i\, \partial_i$ where i runs of the set of indices labeling the base vectors, and analogous statement hold for $(\epsilon_j\,\partial_j)$ and $(\epsilon_k\,\partial_k)$.

Remark: $\partial_j\Phi(\mathbf{x}_0)$ should hence be interpreted as $\partial_j\Phi(\mathbf{x}_0) = \left.\frac{\partial}{\partial x_j}\Phi(x_1,\dots,x_j,\dots)\right|_{\mathbf{x}=\mathbf{x}_0}$.

a) Verify that for scalar arguments $x \in \mathbb{R}$ the expression for the multi-dimensional Taylor expansion reduces to the one for real functions that we have discussed before.

b) Show that the first terms of the Taylor expansion can also be written in the form

$$\Phi(\mathbf{x}) = \Phi(\mathbf{x}_0) + (\boldsymbol{\varepsilon}\cdot\nabla)\Phi(\mathbf{x}_0) + \frac{1}{2}\,\boldsymbol{\varepsilon}^T\,\mathsf{C}(\mathbf{x}_0)\,\boldsymbol{\varepsilon} + \dots$$

where the matrix $\mathsf{C}(\mathbf{x}_0)$ has the components $c_{ij}(\mathbf{x}_0) = \partial_i\partial_j\Phi(\mathbf{x}_0)$.

c) We say that the function $\Phi(\mathbf{x})$ has an extremum at $\mathbf{x}_0$ when $\nabla\Phi(\mathbf{x}_0) = \mathbf{0}$.
Why is this a reasonable based on the special case where Φ depends only on a scalar argument?
What does this imply for the forces acting at the position $\mathbf{x}_0$ when Φ is interpreted as a potential?

d) Verify that the left and the right eigenvectors of C are identical, up to transposition.
Why does this imply that the normalized eigenvectors span a orthonormal basis?

e) Show that $\Phi(\mathbf{x})$ has a minimum at $\mathbf{x}_0$ iff

- $\nabla\Phi(\mathbf{x}_0) = \mathbf{0}$, and
- all eigenvalues of $\mathsf{C}(\mathbf{x}_0)$ are positive.

Also provide the condition for a maximum.
What happens when there are positive and negative eigenvalues?
What does it imply when (some) eigenvalues vanish?

f) Consider now the two-dimensional case $\mathbf{x} \in \mathbb{R}$. Sketch the contour lines of the potential for the following situations

- $\nabla\Phi(\mathbf{x}) = (1,1)$ and $\mathsf{C}(\mathbf{x}) = \mathsf{0}$ for all positions $\mathbf{x}$.
- $\nabla\Phi(1,2) = \mathbf{0}$ and $\mathsf{C}(1,2) = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ with

1. $b > 1$,
2. $1 > b > 1$,
3. $b < -1$,
4. $b = 1$.

Problem 8.3. Mechanical similarity

Two solutions of a differential equations are called *similar* when they can be transformed into one another by a rescaling of the time-, length- and mass-scales. We indicate the rescaled quantities by a prime, and denote the scale factors as τ, λ, and μ, respectively,

$$t' = \tau t \,, \qquad \mathbf{q}_i' = \lambda \mathbf{q}_i \,, \qquad m_i' = \mu m_i$$

a) We consider a system with kinetic energy $T = \frac{1}{2} \sum_i m_i \, \dot{\mathbf{q}}_i^2$, and consider a potential that admits the following scaling

$$V' = \mu^\alpha \, \lambda^\beta \, V$$

Show that the EOM are then invariant when one rescales time as

$$\tau = \mu^{(1-\alpha)/2} \, \lambda^{(2-\beta)/2}$$

Remark: Assume energy conservation for the discussion of the EOM.

b) Consider now two pendulums, $V = mgz$ with different masses and length of the pendulum arms. Which factors τ, λ, and μ relate their trajectories? How will the periods of the pendulums thus be related to the ratio of the mass and the length of the arms? Which scaling do you expect based on a dimensional analysis?

c) What do you find for the according discussion of the periods of a mass attached to a spring, $V = k \, |\mathbf{q}|^2/2$?

d) Discuss the period of the trajectories in the Kepler problem, $V = mMG/|\mathbf{q}|$. In this case the dimensional analysis is tricky because the masses of the sun and of the planet appear in the problem. What does the similarity analysis reveal about the relevance of the mass of the planet for Kepler's third law?