Homework Exercises 7

Chapter 4 of my lecture notes provides the background to solve the following exercises. A PDF-file of your solution to the problems 7.1 – 7.3 should uploaded to your Moodle account

by Sunday, December 20 (with a grace time till christmas).

The parts marked by $\star$ are suggestions for further exploration that will be followed up in the seminars.

Problem 7.2 has been given in this form in a previous exam.

Problems

Problem 7.1. Solving ODEs by separation of variables

Determine the solutions of the following ODEs

a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 y}{\sin^2 x}$ such that $y(\pi/4) = 0$

b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\,x^2\,y}{2y^2+1}$ such that $y(0) = 1$

c) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1+y^3}{x\,y^2\,(1+x^2)}$ such that $y(1) = 2$

Hint: Show and use that $\frac{1}{x\,(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

Problem 7.2. Vectors, derivatives and phase-space portraits

a) Contour lines in (q,p) are lines where a funktion $f(q,p)$ takes a constant value. Sketch the contour lines of

$$f(q,p) = \frac{p^2}{2} - \sin q\,,$$

in order to get an idea about the height profile of this function.

b) A vector field $\mathbf{K}(x, y)$ is conservative if it can be written as a gradient of a potential $U(x, y)$. Which of the following vector fields are conservative:

$$\mathbf{K}_1(x, y) = (x + y, x + y)$$

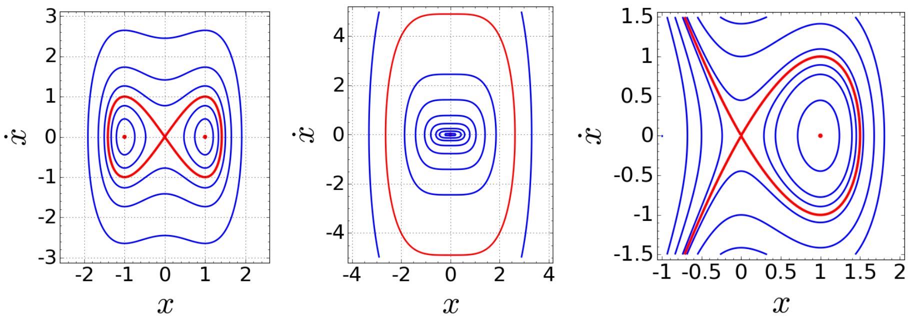
$$\mathbf{K}_2(x, y) = (x - y, x + y)$$

$$\mathbf{K}_3(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

Justify your answer!
Sketch the contour lines and the gradients of the potential for a case where the field is conservative.

c) The graphs below show phase portraits of differential equations of the form

$$\ddot{x}(t) = a\,x + b\,x^2 + c\,x^3 \qquad \text{with} \quad a, b, c \in \mathbb{R}$$



Discuss whether the respective constants a, b and c are positive, negative or whether they vanish.

d) Interpret x the the position of a particle in a potential, and sketch the potentials that result in the given phase portraits.

Problem 7.3. Light intensity at single-slit diffraction

Monochromatic light of wave length λ that is passion through a slit will produce an diffraction pattern on a screen where the intensity follows (cf. Figure 1, left panel)

$$I(x) = I_{\max} \left(\frac{\sin x}{x} \right)^2$$

Here the light intensity $I(x)$ is the power per unit area that is observed at a distance x to the side from the direction staight ahead from the light source through the slit to the screen. We are interested in the total power $P(\Delta)$ that falls into a region of width $|x| < \Delta$. Since there is no antiderivative for $I(x)$ we will find approximate solutions by considering Taylor appoximations of $I(x)$ that can be integrated without effort.

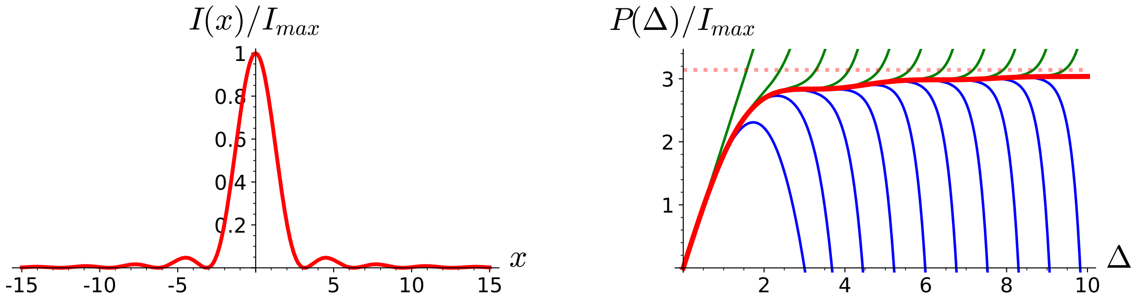


Figure 1: The left panel shows $I(x)/I_{\text{max}}$, and the thick red line in the right panel shows the (normalized) light intensity $P(\Delta) = \left[\int_{-\Delta}^{\Delta} I(x)\mathrm{d}x\right] / I_{\text{max}}$ impinging on a strip of width Δ. The red dotted lines mark the asymptotic value π. The other lines show Taylor approximations or order $n = N$ of the function $P(\Delta)$: green for even orders $N = 0, 2, \dots 20$, and blue for odd orders $N = 1, 3, \dots 19$.

a) Show that $\sin^2 x = (1 - \cos 2x)/2$, and use the Taylor expansion of the cosine-function to show that

$$\frac{\sin^2 x}{x^2} = \frac{1 - \cos 2x}{2x^2} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} (2x)^{2n}$$

b) Determine the Taylor approximations for $P(\Delta)$ by integrating the expression found in a).

⋆ c) Write a program that numerically determines $P(\Delta)$, and compares it to Taylor approximations of different order, as shown in the right panel of Figure 1.

Self Test

Problem 7.4. Taylor series of elementary functions

In the following I provide the first terms of the Taylor expansion for small x of some elementary functions. Verify the expressions by determining the derivatives of the functions,

a) $(1+x)^n \approx 1 + n\,x$

b) $\sqrt{1+x} \approx 1 + \frac{x}{2}$

c) $(1+x)^{-1/2} \approx 1 - \frac{x}{2}$

d) $\sin x \approx x - \frac{x^3}{3!}$

e) $\cos x \approx 1 - \frac{x^2}{2!}$

f) $\tan x \approx x + \frac{x^3}{3}$

g) $\mathrm{e}^x \approx 1 + x$

h) $\ln(1+x) \approx x - \frac{x^2}{2}$

Determine the subsequent order of the Taylor expansion for the following functions. For the first function the result is given as an example:

i) $\frac{1}{1-x} \approx 1 + x + x^2$

j) $(1+x)^n \approx 1 + n\,x + \underline{\qquad}$

k) $\sqrt{1+x} \approx 1 + \frac{x}{2} + \underline{\qquad}$

l) $(1+x)^{-1/2} \approx 1 - \frac{x}{2} + \underline{\qquad}$

m) $\sin x \approx x - \frac{x^3}{3!} + \underline{\qquad}$

n) $\cos x \approx 1 - \frac{x^2}{2!} + \underline{\qquad}$

o) $\tan x \approx x + \frac{x^3}{3} + \underline{\qquad}$

p) $\mathrm{e}^x \approx 1 + x + \underline{\qquad}$

q) $\ln(1+x) \approx x - \frac{x^2}{2} + \underline{\qquad}$

⋆ r) $(1+x)^{1+x} \approx 1 + x + \underline{\qquad}$

Problem 7.5. Substitution with trigonometric and hyperbolic functions

Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{q(x_1)}^{q(x_2)} \mathrm{d}q\; f(q) = \int_{x_1}^{x_2} \mathrm{d}x\; q'(x)\; f(q(x))$$

with a function $q(x)$ that is bijective on the integration interval $[x_1, x_2]$.

a) $\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}}$ by substituting $x = \sin\theta$

b) $\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}}$ by substituting $x = \sinh z$

c) $\int_a^b \mathrm{d}x\, \frac{1}{1+x^2}$ by substituting $x = \tan\theta$

d) $\int_a^b \mathrm{d}x\, \frac{1}{1-x^2}$ by substituting $x = \tanh z$

Problem 7.6. Volume and surface of solids of revolution

The surface of a solid of revolution can be obtaned by rotating some function $f(x)$ around the x axis. For instance, the function $\sqrt{R^2 - x^2}$ with $-R \leq x \leq R$ describes a sphere of radius R. The volume V and the surface O of a solid of revolution are given by the integrals

$$V = \pi \int \mathrm{d}x\ (f(x))^2 \qquad O = 2\,\pi \int \mathrm{d}x\, f(x)\,\sqrt{1 + \big(f'(x)\big)^2}$$

a) Sketch the function $f(x) = \sqrt{R^2 - x^2}$ and verify the the solid of revolution is indeed a sphere.

b) Determine the volume and the surface of the sphere by evaluating the integrals provided above.

Bonus Problem

Problem 7.7. Can you paint this funnel?

We consider the solid of revolution that is described by $f(x) = x^{-1}$ with $x \geq 1$.

a) Sketch the function $f(x)$. It describes a funnel of infinite length, or a "horn".

b) Show that the funnel has the volume π.

c) Determine the surface $O(L)$ of the piece of the funnel with $x \in \{1, L\}$. How does the function $O(L)$ look like in the limit $L \to \infty$? What does this imply for the surface area of the funnel?

d) Since the surface area of the funnel is infinite, one would expect that it can not painted with a finite amount of paint. On the other hand, it volume is finite such that it can be filled with a finite amount of paint. Wouldn't that imply that the surface is painted? How can that be?