Homework Exercises 6

Chapter 4 of my lecture notes provides the background to solve the following exercises. A PDF-file of your solution to the problems 6.2 and 6.3 should uploaded to your Moodle account

by Sunday, Nov 13 (with a grace time till Monday morning).

The Moodle problem 6.1 will be available by Friday.

The parts marked by $\star$ are suggestions for further exploration that will be followed up in the seminars. Additional self-test problems and a bonus problem will be added in the course of the week.

Problems

Problem 6.1. Solutions of ODEs

This will be a Moodle test that will be performed and evaluated on the Moodle platform.

Problem 6.2. Damped oscillator in phase space

We consider the EOM

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = -\frac{k}{m}\, x(t) - \gamma\, v(t)$$

that describes the damped motion of a particle of mass m that is attached to a spring with spring constant k.

a) What are the SI units of the spring constant k and the damping constant γ?

b) When a particle is placed, with zero velocity, at a rest posion x_0, then it will remain at the rest position. In other words, the rest position $(x_0, 0)$ is a fixed point of the EOM. Determine the rest position of the particle at the spring.

c) Let $E = \frac{m}{2}\,\dot{x}^2(t) + \frac{k}{2}\,x^2(t)$ be the energy of the particle.
Why is this a meaningful choice of the energy?
Show that $\frac{\mathrm{d}E}{\mathrm{d}t} = -m\,\gamma\,\dot{x}^2(t)$.

d) We start the oscillator with energy E_0. Show that $A = \sqrt{E_0/k}$ is a length scale and that $T = \sqrt{m/k}$ is a time scale.
Consider now the dimensionless position $\xi = x/A$, velocity $\zeta = vT/A$, energy $\mathcal{E} = E/E_0$, and time $\tau = t/T$. Show that

$$\dot{\mathcal{E}} = -c\,\zeta^2\,,$$

How does the constant c depend on the parameters of the EOM: m, k, and γ. (This should be a result of your derivation of the equation.)
Bonus: How would you determine c by dimensional analysis?

e) Sketch the evolution of $\big(\xi(\tau), \zeta(\tau)\big)$ in phase space when $\gamma = 0$?
How is the distance from the origin related to the energy of the oscillator? What does this imply for the admissible initial positions of the trajectory in phase space?
The time derivative of $\mathcal{E}$ is strictly negative when $\gamma > 0$. What does this imply for the evolution of the trajectory in phase space? Where will it end for large times $\tau \gg 1$?

⋆ f) Assume that the solution of the EOM takes the form

$$x(t) = x_0\,\sin(\omega t - \varphi)\,\mathrm{e}^{-t/t_c}$$

How should ω and t_c be related in that case to k, m, und γ?
Is there also a condition on φ?
Bonus: Can you answer the question concerning φ *without* performing a calculation?

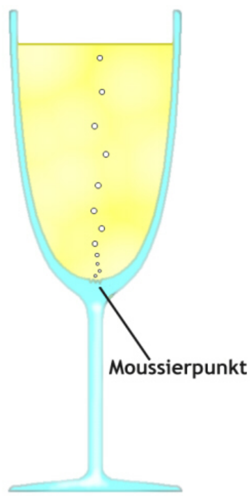
Problem 6.3. Bubbles rising in a fluid

Champaign owns its esprit from the sparkling of the rising air bubbles. When they arrive at the surface they burst and release odor and flavor. During its rise a gas bubble of radius R experiences an upward buoyant force due to Archimedes' principle

$$F_g = \frac{4\,\pi}{3}\,R^3\,(\varrho_f - \varrho_g)\,g$$

where ϱ_g and ϱ_f are the mass density of the gas and the surrounding fluid. Moreover, when it rises with a speed $\dot{z}(t)$ it also experiences a Stokes friction force

$$F_S = -6\,\pi\,\eta\,R\,\dot{z}(t)$$



a) Determine the EOM for the height of $z(t)$ of the gas bubble in the glass, and write the equation in the form

$$I = \ddot{z}(t) + c_1\, \dot{z}(t) + c_0\, z(t)\,.$$

How do the constant coefficients I, c_1 and c_0 depend on the parameters of the forces?

b) Determine the length and time scale, and a constant reference velocity such that the dimensionless position ξ of the bubble, and its dimensionless velocity $\zeta = \mathrm{d}\xi/\mathrm{d}\tau$ are described by the following homogeneous ODE

$$0 = \ddot{\xi}(\tau) + \dot{\xi}(\tau) \tag{6.1}$$

Hint: The choice of a constant reference velocity implies here that one selects the coordinate system that moves upwards with a constant velocity such that the bubble is at rest in this reference system at late times.

c) Sketch the velocity field in the phase space and some solutions of Equation (6.1).

d) Determine the solution of the dimensionless velocity $\zeta(\tau)$ and the position $\xi(\tau)$.

e) Determine $z(t)$ by substituting the definitions of the dimensionless unites in the result for $\xi(\tau)$.

⋆ f) Compare your result to Eq. (4.3.3b) of my lecture notes.
What is the relation between these expressions?