Homework Exercises 5

Chapter 3 of my lecture notes provides the background to solve the following exercises. A PDF-file of your solution to the problems 5.1–5.4 should uploaded to your Moodle account

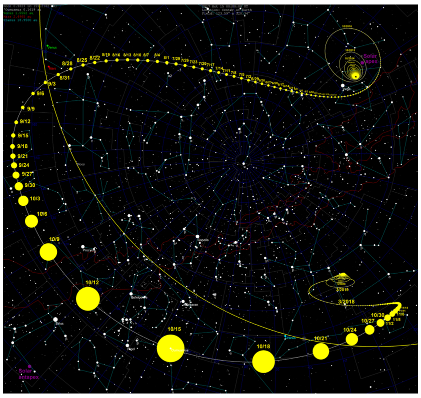
by Sunday, Nov 6 (with a grace time till Monday morning).

The parts marked by ⋆ are suggestions for further exploration that will be followed up in the seminars.

In the self-test exercises you can check your understanding of the consequences of forces and acceleration (Problem 5.5), and explore the physics of elastic collisions (Problem 5.6). The bonus problem is a teaser that illustrates energy transfer in elastic collisions.

Problems

Problem 5.1. ’Oumuamua



On 19 October 2017 astronomers at the Haleakala Observatory in Hawaii discovered ’Oumuamua, the first interstellar object observed in our solar system. It approached the solar system with a speed of about $v_I = 26\,\text{km/s}$ and reached a maximum speed of $v_P = 87.71\,\text{km/s}$ at its perihelion, i.e., upon closest approach to the sun on 9 September 2017.

a) Show that at the perihelion the speed and ’Oumuamua’s smallest distance to the sun, D, obey the relation[1]

$$\frac{v_P^2 - v_I^2}{2} = \frac{M_S\, G}{D}$$

[1]Hint: Consider energy conservation and note that the potential energy of a body of mass m at a distance D from the sun is $\Phi(D) = m\, M_S\, G/D$.

while for the Earth we always have[2]

$$\frac{4\pi^2 R}{T^2} \simeq \frac{M_S\, G}{R^2}$$

Here, M_S is the mass of Sun, R is the Earth-Sun distance, and $T = 1\,\text{year}$ is the period of Earth around Sun.

b) Show that this entails that $\dfrac{D}{R} = \dfrac{2\, v_E^2}{v_P^2 - v_I^2}$, where $v_E = 2\pi R/T$ is the speed of Earth around sun.

c) Use the relation obtained in (b) to determine D in astronomical units, and compare your estimate with the observed value $D = 0.25534(7)\text{AU}$.

Problem 5.2. Motion in a harmonic central force field

A particle of mass m and at position $\mathbf{r}(t)$ is moving under the influence of a central force field

$$\mathbf{F}(\mathbf{r}) = -k\, \mathbf{r}\, .$$

a) What are the units of the scalar constant k?

b) We want to use the force to build a particle trap,[3] i.e. to make sure that the particle trajectories $\mathbf{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\mathbf{r}(t)| < B$ for all times t. What is the requirement on the values of k to achieve this aim?

c) Determine the energy of the particle and show that the energy is conserved.
Hint: When energy is conserved it can be written as a sum of kinetic and potential energy. How would you determine the potential energy in this case?

d) Demonstrate that the angular momentum $\mathbf{L} = \mathbf{r} \times m\, \dot{\mathbf{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field does no longer coincide with the origin of the coordinate system in that case, i.e. $\mathbf{F}(\mathbf{r}(t)) = -k\,(\mathbf{r}(t) - \mathbf{r}_\text{o})$ where the constant vector $\mathbf{r}_\text{o}$ describes the position of the center of the force field.

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[2]Hint: Assume that the Earth orbit is circular, and explore Newton's 2nd law.

[3]Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

Problem 5.3. Forces, potentials, and line integrals

Consider the force $\mathbf{F}(x, y, z) = (x + yz, y + xz, z + xy)$.

a) Evaluate the line integral $\int_{\gamma_1} \mathrm{d}\mathbf{s} \cdot \mathbf{F}$ for the path $\gamma_1 = (at, bt, ct)$ with some real constants a, b, c and $t \in [0, 1]$.

b) Evaluate also the line integral $\int_{\gamma_2} \mathrm{d}\mathbf{s} \cdot \mathbf{F}$ for the path γ_2 that comprises the three pieces $(0, 0, 0) \to (a, 0, 0)$ along the x axis, then $(a, 0, 0) \to (a, b, 0)$ parallel to the y-axis, and finally $(a, b, 0) \to (a, b, c)$ parallel to the z-axis.

c) Determine $\nabla \times \mathbf{F}$.

d) Is the force $\mathbf{F}$ conservative? Substantiate your reply!
If no: Why not?
If yes: Determine the potential $\Phi(x, y, z)$ with $\mathbf{F}(x, y, z) = -\nabla\Phi(x, y, z)$.

Problem 5.4. Inelastic collisions, ballistics, and cinema heroes

We first discuss a few CSI techniques to investigate firearms. Then we wonder how cinema heroes shoot.

a) The velocity of a projectile can be determined by investigating its impact into a wooden block (mass M) that is fixed to a rotor arm that lets it move horizontally on a circular track with radius ℓ. We choose our coordinates such that the block moves in the $(x - y)$-plane and that the rotor axis is at the origin of the coordinate system. The angle θ describes the angle of the arm with respect to the positive x axis.

   We set up the experiment such that initially $\theta = 0$ while the projectile approaches the wooden block along a trajectory parallel to the y-axis. At time t_0 it hits the center of the block. The projectile has mass m and velocity $\mathbf{v}_0$. Sketch the setup.

b) What is the angular momentum of the projectile, the wooden block and the total angular momentum before the impact of the bullet? How does it change upon impact? What is the angular momentum after the impact? What is the angular speed of the rotor after the collision? What does this tell about the speed of the projectile?

c) Determine the kinetic energy of the projectile and of the rotor after the impact. What might be the origin of the energy difference?

d) The title of Stanley Kubrick's movie *Full Metal Jacket* refers to full metal jacket bullets, i.e. projectiles as they were used in the M16 assault rifle used in the Vietnam war. Its bullets have a mass of 10 g and they set a 1 kg wooden block revolving at a 1 m arm into a 8 Hz motion. What is the velocity of the bullets?

e) Alternatively one can preform this measurement by shooting the bullet into a swing where a wooded block of mass M is attached to ropes of length L. Initially it is at rest. Consider momentum conservation to determine its velocity immediately after impact. What does this tell about the kinetic energy immediately after the impact, and what about the maximum height of reached by the swing in its subsequent motion?

   Let L be 0.2 m. Which mass is required to let the swing go up to the height of its spindle?

⋆ f) In the continuous fire mode the M16 has a rate of fire of 700–950 rounds/min. What does this tell about the recoil of the rifle? What do you think now about the Rambo shooting scene that you can find here on YouTube?

Self Test

Problem 5.5. Car on a air-cushion

We consider a car of mass $m = 20\,\mathrm{g}$ moving – to a very good approximation without friction – on an air-cushion track. There is a string attached to the car that moves over a roll and hangs vertically down on the side opposite to the car.

a) Sketch the setup and the relevant parameters.

b) Which acceleration is acting on the car when the string is vertically pulled down with a force of $F = 2\,\mathrm{N}$. Determine the velocity $v(t)$ and its position $x(t)$.

c) Determine the force acting on a 200 g chocolate bar, in order to get a feeling for the size of the force that was considered in (b).

d) Now we fix the chocolate bar at the other side of the string. The equation can then be obtained based on energy conservation

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{m+M}{2}\, v^2 + Mgh = \text{konst},$$

where M is the mass of the chocolate bar. Is the acceleration the same of different as in the cases (b) and (c)? Provide an argument for your conclusion.

Problem 5.6. Collision with an elastic bumper

Consider two balls of radius R with masses m_1 and m_2 that are moving along a line. Their positions will be denoted as x_1 and x_2 in such a way that they touch when $x_1 = x_2$ and they do not feel each other when $x_1 < x_2$. When they run into each other, the balls can slightly be deformed such that the distance between their centers takes the value $2R - d$, and they experience a harmonic repulsive forces $\pm k\, d$. We will say then that $d = x_2 - x_1 < 0$.

a) During the collision of the two balls Newton's equations take the form

$$m_1\, \ddot{x}_1(t) = -k\, d(t) \qquad\qquad m_2\, \ddot{x}_2(t) = k\, d(t)$$

Show that this implies

$$\ddot{d} = -\omega^2\, d$$

for some positive constant ω. How does ω depend on the spring constant k and on the masses m_1 and m_2?

b) Let $d(t) = -d_M\, \sin\big(\omega\,(t - t_0)\big)$ describe the deformation of the balls for a collision at $t = t_0$, and contact in the time interval $t_0 \le t \le t_R$. Verify that it is a solution of the equation of motion. At which time t_R will the particles release (i.e. there is no overlap any longer)? What is the maximum potential energy stored in the harmonic potential?

c) We consider initial conditions where particle 1 arrives with a constant velocity v_0 from the left, and particle 2 is at rest. What is the total kinetic energy in this situation? Assume that at most a fraction α of the kinetic energy is transferred to potential energy. What is the relation between v_0 and the maximum deformation d_M?

d) The velocity of the two particles at times $t_0 \leq t \leq t_R$ can now be obtained by solving the integrals

$$m_i\, \dot{x}_i(t) = m_i\, x_i(t_0) + (-1)^i \int_{t_0}^{t} \mathrm{d}t'\, k\, d(t')\,, \quad \text{with } i \in \{1, 2\}$$

Why does this hold? Which values does $x_i(t_0)$ take? Solve the integral and show that

$$\dot{x}_1 = v_0\, \Big[1 + \sqrt{\alpha\beta}\, \Big(\cos(\omega\,(t - t_0)) - 1\Big)\Big]$$
$$\dot{x}_2 = v_0\, \frac{m_1}{m_2}\, \sqrt{\alpha\beta}\, \Big(\cos(\omega\,(t - t_0)) - 1\Big)$$

How does β depend on the masses?

e) Verify that at release we have

$$\dot{x}_1 = v_0\, \big(1 - 2\,\sqrt{\alpha\beta}\big)$$
$$\dot{x}_2 = v_0\, \frac{2\, m_1}{m_2}\, \sqrt{\alpha\beta}\big)$$

Verify that these expressions comply to momentum conservation. Verify that the expressions obey energy conservation iff $\alpha = \beta = m_2/(m_1 + m_2)$.

f) What does this imply for particles of identical masses, $m_1 = m_2$? How does your result fit to the motion observed in Newton's cradle? What does it tell about the assumption of instantaneous collisions of balls that is frequently adopted in theoretical physics?

Bonus Problem

Problem 5.7. Galilean cannon

In the margin we show a sketch of a Galilean cannon. Assume that the mass mass ratio of neighboring balls with always two, and that they perform elastic collisions.

a) Initially they are stacked exactly vertically such that their distance is negligible. Let the distance between the ground the lowermost ball be 1 m. How will the distance of the balls evolve prior to the collision of the lowermost ball with the ground?

b) After the collision with the ground the balls will move up again. Determine the maximum height that is reached by each of the balls.