Homework Exercises 4

Chapter 3 of my lecture notes provides the background to solve the following exercises. A PDF-file of your solution to the problems 4.2–4.3 should uploaded

to your Moodle account

by Sunday, Nov 29 (with a grace time till Monday morning)

The parts marked by $\star$ are suggestions for further exploration that will be followed up in the seminars, but need not be submitted and will not be graded. In the self-test exercises you can check you background and understanding of forces, vectors, and coordinates. However, the parts marked by might take some extra effort to solve. The bonus problem explores the relation between systems of linear equations, their solutions, and the properties of vector spaces.

Problems

Problem 4.1. Derivatives of Common Composite Expressions
This will be a Moodle test that will be performed and evaluated on the Moodle platform.

Problem 4.2. Contour lines and gradients

a) Determine the gradients of the functions

$$A(\mathbf{q}) = A_0 \sin(\mathbf{q} \cdot \mathbf{q}) \qquad\qquad B(\mathbf{q}) = B_0 \sin(\mathbf{k} \cdot \mathbf{q})$$

where $\mathbf{k} \in \mathbb{R}^2$ is a constant vector and $\mathbf{q} = (x, y)$ a position in $\mathbb{R}^2$.

b) Determine the lines $y_c(x)$ where the functions take constant values. On a map these are contour lines. For a potential these are equipotential lines.

c) Show that the gradient is oriented vertically to the contour lines.
Hint: Consider a trajectory $\mathbf{q}_c(t) = \big(x(t), y_c(x(t))\big)$, and argue
1) that it describes the motion along a contour line, and
2) that $\dot{\mathbf{q}}_c(t)$ is therefore aligned parallel to the contour line.

d) Sketch the contour lines of the functions, and mark the gradients by arrows.

Problem 4.3. Dynamics in phase space

We consider the following equations of motion

$$\dot{x}(t) = \alpha\, x(t)\, \big(p(t) - 1\big)\,, \qquad \dot{p}(t) = \quad p(t)\, \big(1 - x(t)\big)$$

where α is a positive real number, and $x(t)$, $p(t)$ are real-valued functions.

a) Determine the fixed points of the dynamics,
i.e. points (x_0, p_0) where $\big(\dot{x}(t),\, \dot{p}(t)\big)_{(x(t),\,p(t))=(x_0,p_0)} = \mathbf{0}$.

b) Show that

$$I = x(t) + \alpha\, p(t) - \ln\big(x(t)\, p^{\alpha}(t)\big)$$

is a constant of motion of the dynamics, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} I = 0$.

* c) What is the relation between the contour lines of I, and the solution of the equations of motion stated in the beginning of the problem?

** d) For small ϵ one has $1 + \epsilon - \ln(1+\epsilon) \simeq 1 + \epsilon^2/2$. Use this information to sketch the trajectories $(x(t), p(t))$ in the phase space, i.e. in the (x,p)-plain.

Self Test

Problem 4.4. Pulling a duck

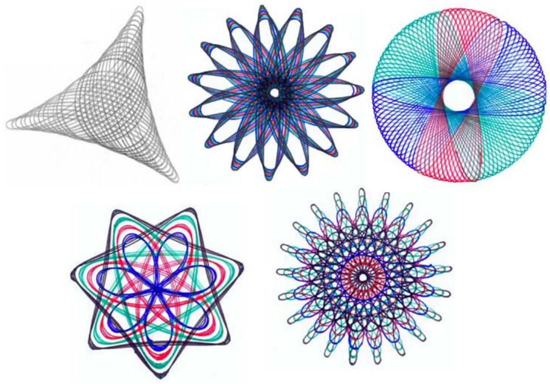
A child is pulling a toy duck with a force of $F = 5\,\mathrm{N}$. The duck has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.[1]

a) Describe the motion of the duck when there is no friction.
In the beginning the duck is at rest.

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the duck.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

Problem 4.5. Hypotrochoids, roulettes, and the spirograph


[Wikimedia Public domain]

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside
or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

a) Consider the track of a pole attached to a disk with n cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the curves developed in Problem 3.3 such that you can describe hypotrochoids.

c) Test your result by writing a program that plots the curves for given m and n.
Remark: More explanation and a Sage Notebook that you can use to start this analysis will be given in the wiki.

[1]For this angle one has $\tan\theta \approx 3/4$.

Bonus Problem

Problem 4.6. Maximum distance of flight

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

Challenge. What is the optimum choice of θ for the shot put?

c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?