Homework Exercises 3

Chapter 2 of my lecture notes provides the background to solve the following exercises. Your solution to the problems 3.1–3.3 should uploaded

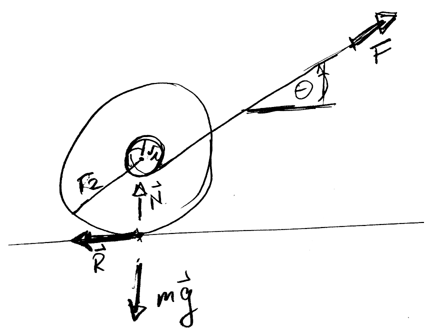
to your Moodle account
as a PDF-file
by Sunday, Nov 22 (with a grace time till Monday morning)

The parts marked by ⋆ are suggestions for further exploration that will be followed up in the seminars, but need not be submitted and will not be graded. In the self-test exercises you can check you background and understanding of forces, vectors, and coordinates. However, the parts marked by ☡ might take some extra effort to solve. The bonus problem explores the relation between systems of linear equations, their solutions, and the properties of vector spaces.

Problems

Problem 3.1. Walking a yoyo

The sketch to the right shows a yoyo of mass m standing on the ground. It is held at a chord that extends to the top right. There are four forces acting on the yoyo: gravity $m\mathbf{g}$, a normal force $\mathbf{N}$ from the ground, a friction force $\mathbf{R}$ at the contact to the ground, and the force $\mathbf{F}$ due to the chord. The chord is wrapped around an axle of radius r_1. The outer radius of the yoyo is r_2.



a) Which conditions must hold such that there is no net force acting on the center of mass of the yoyo?

b) For which angle θ will the torque vanish?

⋆ c) Perform an experiment: What happens for larger and for smaller angels θ? How does the yoyo respond when you fix the height where you keep the chord and pull continuously?

Problem 3.2. Properties of the cross product

Verify the following properties of the cross product:

a) Two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are linearly independent unless their cross product vanishes

$$\forall \lambda \in \mathbb{R}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 : \quad \mathbf{b} = \lambda \mathbf{a} \Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}\,.$$

b) When three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are linearly dependent, i.e. when one of them can be expressed as a linear combination of the other two, then their scalar triple product vanishes

$$\forall \alpha, \beta \in \mathbb{R}, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 : \quad \mathbf{c} = \alpha \mathbf{a} + \beta \mathbf{b} \Rightarrow (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0\,.$$

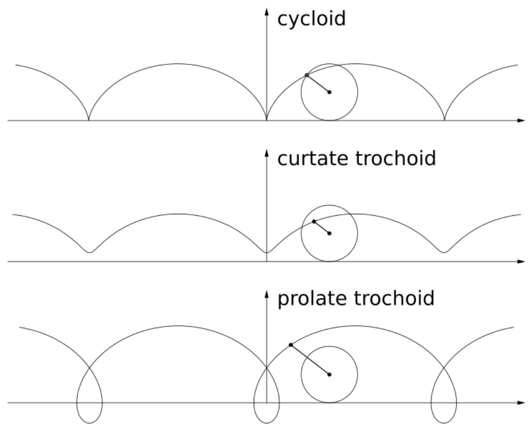
What must be adapted in this agrument when $\mathbf{a}$ is a linear combination of $\mathbf{b}$ and $\mathbf{c}$?

c) All vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ obey the identity:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\,.$$

Problem 3.3. Retroreflector paths on bike wheels

The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retroreflectors fixed in the sparks enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio ρ of the reflectors distance d to the wheel axis and the wheel radius r.



[Wikimedia CC BY 4.0 (modified)]

A small stone in the profile traces a cycloid ($\rho = 1$). Animations of the trajectories are provided in the Wiki. It also provides a Jupyter Notebook that allows you to explore the parameter dependence of the tracks.

A trochoid is most easily described in two steps: Let $\mathbf{M}(\theta)$ be the position of the center of the disk, and $\mathbf{D}(\theta)$ the vector from the center to the position $\mathbf{q}(\theta)$ that we follow (i.e. the position of the retroreflector) such that $\mathbf{q}(\theta) = \mathbf{M}(\theta) + \mathbf{D}(\theta)$.

a) The point of contact of the wheel with the street at the initial time t_0 is the origin of the coordinate system. Moreover, we single out one spark and denote the change of its angle with respect to its initial position as θ. Note that negative angles θ describe forward motion of the wheel!

   Sketch the setup and show that

$$\mathbf{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix}, \qquad \mathbf{D}(\theta) = \begin{pmatrix} -d\,\sin(\varphi+\theta) \\ d\,\cos(\varphi+\theta) \end{pmatrix}.$$

   What is the meaning of φ in this equation?

b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time, $L = \int_{t_0}^{t} \mathrm{d}t\; |\dot{\mathbf{q}}(\theta(t))|$. Show that therefore

$$L = r \int_0^\theta \mathrm{d}\theta\, \sqrt{1 + \rho^2 + 2\rho\,\cos(\varphi+\theta)}$$

c) Consider now the case of a cycloid and use $\cos(2x) = \cos^2 x - \sin^2 x$ to show that the expression for L can then be written as

$$L = 2\,r \int_0^\theta \mathrm{d}\theta\, \left|\cos\frac{\varphi+\theta}{2}\right|$$

   How long is one period of the track traced out by a stone picked up by the wheel profile?

Self Test

Problem 3.4. Euler's equation and trigonometric relations

Euler's equation $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\,\sin x$ relates complex values exponential functions and trigonometric functions.

a) Sketch the position of $\mathrm{e}^{\mathrm{i}x}$ in the complex plain, and indicate how Euler's equation is related to the Theorem of Pythagoras.

b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular one has $e^{i(x+y)} = e^{ix} e^{iy}$. Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about $\sin(x+y)$ and $\cos(x+y)$?

c) Simplify the general expressions for the special cases $\sin(2x)$ and $\cos(2x)$.

Problem 3.5. Crossing a river

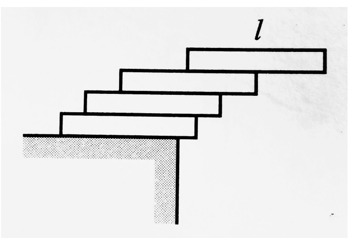
A ferry is towed at the bank of a river of width $B = 100$ m that is flowing at a velocity $v_F = 4$ m/s to the right. At time $t = 0$ s it departs and is heading with a constant velocity $v_B = 10$ km/h to the opposite bank.

a) When will it arrive at the other bank when it always heads straight to the other side? (In other words, at any time its velocity is perpendicular to the river bank.) How far will it drift downstream on its journey?
Hint: Describe its position as the sum of two vectors: one accounts for the motion of boat in the direction to the opposite bank — the other for the effect of the drift off due to the riven.

b) In which direction (i.e. angle of velocity relative to the downstream velocity of the river) must the ferryman head to reach exactly at the opposite side of the river? Determine first the general solution. What happens when you try to evaluate it for the given velocities?

Bonus Problem

Problem 3.6. Piling bricks



Christmas is approaching, and Germans consume enormous amounts of chocolate. If you happen to come across a considerable pile of chocolate bars (or beer mats, or books, or anything else of that form) I recommend the following experiment:

a) We consider N bars of length l piled on a table. What is the maximum amount that the topmost bar can reach beyond the edge of the table.

b) The sketch above shows the special case $N = 4$.
However, what about the limit $N \to \infty$?