Homework Exercises 2

Chapter 2.5–2.8 of my lecture notes provide the background to solve the following exercises. Your solution to the problems 2.1–2.3 should uploaded

to your Moodle account

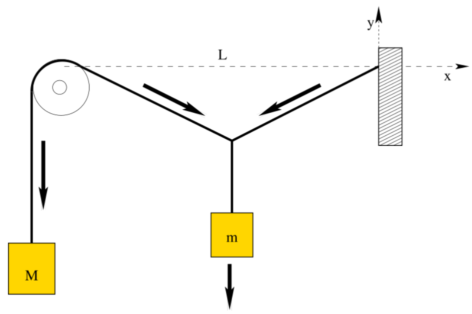
as a PDF-file

by Sunday, Nov 15 (with a grace time till Monday morning)

The parts marked by ⋆ are suggestions for further exploration that will be followed up in the seminars, but need not be submitted and will not be graded. In the self-test exercises you can check you background and understanding of forces, vectors, and coordinates. However, the parts marked by ☡ might take some extra effort to solve. The bonus problem explores the relation between systems of linear equations, their solutions, and the properties of vector spaces.

Problems

Problem 2.1. Mounting street laterns and trolley systems



A latern of mass m is mounted in a wall at the right side of a street, and at the right side one uses a roll and a counterweight of mass M (see sketch). This setup is also widely used for the hanging of cables in trolley systems.

a) Label the forces in the sketch: Denote the weight of the mass M as $\mathbf{F}_M$; the weight of the lamp as $\mathbf{F}_m$; the force along the suspension cable to the left of the lamp as $\mathbf{F}_1$ and the force to the right as $\mathbf{F}_2$.

b) The suspension of the roll is exerting a force $\mathbf{F}_R$ on the roll such that it does not move. How can this force be expressed in terms of the forces introduced in a)? Determine the force graphically and add it to your sketch.

c) Let the masses of the lamp and the counterweight be $m = 15\,\text{kg}$ und $M = 80\,\text{kg}$, respectively. Determine the angle α between the horizontal the suspension cable, when the lamp is positioned right in the middle between the wall and the roll. Determine $|\mathbf{F}_R|$.

d) The angle α is a function of the mass ratio m/M. Why is this not unexpected?

Determine the function $\alpha(m, M)$, and sketch the angle as function of the ratio m/M. What happens when $m > 2M$?

Hint: Try it! The setup can easily build at home with a wire and two weights. Photos and decriptions of measurements of $\alpha(m, M)$ will be published in the wiki, and awarded by bonus points.

$\star$ e) What is the maximum admissible mass when the wall anchor can support a mximum load of 14.0 kN? Whis value does the angle α take in that case?

$\star$ f) Why is it a bad idea to stretch a wire horizontally between two wall anchors, and then use it to support a lamp or some other heavy object? What is the advantage of the trolley sytem with the roll – in particular, when the lamp gains substantial additional weight after a freezing-rain shower?

Problem 2.2. Linear Dependence of three vectors in 2D

In the lecture I pointed out that every vector $\mathbf{v} = (v_1, v_2)$ of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{v} = \alpha\,\mathbf{a} + \beta\,\mathbf{b}$$

In this exercise we revisit this statement for $\mathbb{R}^2$ with the standard forms of vector addition and multiplication by scalars.

a) Provide a triple of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{v}$ such that $\mathbf{v}$ can *not* be represented as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

b) To be specific we will henceforth fix

$$\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers α and β such that

$$\mathbf{v} = \alpha\,\mathbf{a} + \beta\,\mathbf{b}$$

Hint: Verify that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. How would you determine α and β when the vectors $\mathbf{a}$ and $\mathbf{b}$ had unit length? What changes when they have a different length?

c) Consider now also a third vector

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for (α, β, γ) such that

$$\mathbf{v} = \alpha\, \mathbf{a} + \beta\, \mathbf{b} + \gamma \mathbf{c}$$

What is the general constraints on (α, β, γ) such that $\mathbf{v} = \alpha\, \mathbf{a} + \beta\, \mathbf{b} + \gamma \mathbf{c}$. What does this imply on the number of solutions?

d) Discuss now the linear dependence of the vectors **a**, **b** and **c** by exploring the solutions of

$$\mathbf{0} = \alpha\, \mathbf{a} + \beta\, \mathbf{b} + \gamma \mathbf{c}$$

How are the constraints for the null vector related to those obtained in part c)?

Problem 2.3. Polynomials are a vector space

We consider the set of polynomials $\mathbb{P}_N$ of degree N with real coefficients p_k, $k \in \{0, \ldots, N\}$,

$$\mathbb{P}_N := \left\{ \mathbf{p} = \left(\sum_{k=0}^{N} p_k\, x^k \right) \quad \text{with} \;\; p_k \in \mathbb{R},\; k \in \{0, \ldots, N\} \right\}$$

a) Demonstrate that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is a vector space when one adopts the operations

$$\forall \quad \mathbf{p} = \left(\sum_{k=0}^{N} p_k\, x^k \right) \in \mathbb{P}_N\,, \quad \mathbf{q} = \left(\sum_{k=0}^{N} q_k\, x^k \right) \in \mathbb{P}_N\,, \text{ and } \; c \in \mathbb{R} \;:$$

$$\mathbf{p} + \mathbf{q} = \left(\sum_{k=0}^{N} (p_k + q_k)\, x^k \right) \qquad \text{and} \qquad c \cdot \mathbf{p} = \left(\sum_{k=0}^{N} (c\, p_k)\, x^k \right)\,.$$

b) Demonstrate that

$$\mathbf{p} \cdot \mathbf{q} = \left(\int_0^1 \mathrm{d}x \left(\sum_{k=0}^{N} p_k\, x^k \right) \left(\sum_{j=0}^{N} q_j\, x^j \right) \right)\,,$$

establishes an inner product on this vector space.

c) Demonstrate that the three polynomials $\mathbf{b}_0 = (1)$, $\mathbf{b}_1 = (x)$ und $\mathbf{b}_2 = (x^2)$ form a basis of the vector space $\mathbb{P}_2$: For each polynomial $\mathbf{p} \in \mathbb{P}_2$ there are real numbers x_k, $k \in \{0, 1, 2\}$, such that $\mathbf{p} = x_0\,\mathbf{b}_0 + x_1\,\mathbf{b}_1 + x_2\,\mathbf{b}_2$. However, in general we have $x_i \neq \mathbf{p} \cdot \mathbf{b}_i$. Why is that?
Hint: Is this an orthonormal basis?

d) Demonstrate that the three vectors $\hat{e}_0 = (1)$, $\hat{e}_1 = \sqrt{3}\,(2\,x - 1)$ and $\hat{e}_2 = \sqrt{5}\,(6\,x^2 - 6\,x + 1)$ are orthonormal.

e) Demonstrate that every vector $\mathbf{p} \in \mathbb{P}_2$ can be written as a scalar combination of $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$,

$$\mathbf{p} = (\mathbf{p} \cdot \hat{e}_0)\,\hat{e}_0 + (\mathbf{p} \cdot \hat{e}_1)\,\hat{e}_1 + (\mathbf{p} \cdot \hat{e}_2)\,\hat{e}_2\,.$$

Hence, $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$ form an orthonormal basis of $\mathbb{P}_2$.

$\star$ f) Find a constant c and a vector $\hat{n}_1$, such that $\hat{n}_0 = (c\,x)$ and $\hat{n}_1$ form an orthonormal basis of $\mathbb{P}_1$.

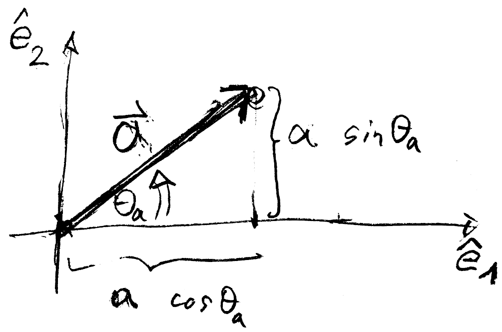
Self Test

Problem 2.4. Geometric and algebraic form of the inner product

The sketch to the right shows a vector $\mathbf{a}$ in the plane, and its representation as a linear combination of two orthonormal vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$,

$$\mathbf{a} = a\,\cos\theta_a\,\hat{\mathbf{e}}_1 + a\,\sin\theta_a\,\hat{\mathbf{e}}_2$$

Here, a is the length of the vector $\mathbf{a}$, and $\theta_1 = \angle(\hat{\mathbf{e}}_1, \mathbf{a})$.



a) Analogously to $\mathbf{a}$ we will consider another vector $\mathbf{b}$ with a representation

$$\mathbf{b} = b\,\cos\theta_b\,\hat{\mathbf{e}}_1 + b\,\sin\theta_b\,\hat{\mathbf{e}}_2$$

Employ the rules of inner products, vector addition and multiplication with scalars to show that

$$\mathbf{a}\cdot\mathbf{b} = a\,b\,\cos(\theta_a - \theta_b)$$

Hint: Work backwards, expressing $\cos(\theta_a - \theta_b)$ in terms of $\cos\theta_a$, $\cos\theta_b$, $\sin\theta_a$, and $\sin\theta_b$.

b) As a shortcut to the explicit calculation of a) one can introduce the coordinates $a_1 = a\,\cos\theta_a$ and $a_2 = a\,\sin\theta_a$, and write $\mathbf{a}$ as a tuple of two numbers. Proceeding analogously for $\mathbf{b}$ one obtains

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

How will the product $\mathbf{a}\cdot\mathbf{b}$ look like in terms of these coordinates?

c) How do the arguments in a) and b) change for D dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_D$?

What changes when the basis vectors are not normalized?
What when they are not even orthogonal?

Problem 2.5. Geometric interpretation of matrices

We explore the set of the eight matrices

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \quad \text{with} \;\; a, b, c, d \in \{\pm 1\} \right\}$$

a) Let the action $\circ$ denotes matrix multiplication. Verify that $(M, \circ)$ is a group. We denote its neutral element as $\mathbb{I}$.

b) Show that the group has five non-trivial elements $s_1, \ldots s_5$ that are self inverse:

$$s_i \neq \mathbb{I} \quad \wedge \quad s_i \circ s_i = \mathbb{I} \quad \text{for} \quad i \in \{1, \ldots 6\}\,.$$

c) Show that the other two elements d and r obey $d \circ r = r \circ d = \mathbb{I}$, that $r = d \circ d \circ d$, and that $d = r \circ r \circ r$.

d) Show that the set of points $P = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$ is mapped to P by the action of an element of the group:

$$\forall m \in M \;\; \wedge \;\; p \in P : \quad p \circ m \in P$$

Hint: The action of the matrix on the vector defined as follows

$$(v_1, v_2) \circ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} v_1\, m_{11} + v_2\, m_{21} \\ v_1\, m_{12} + v_2\, m_{22} \end{pmatrix}$$

e) What is the geometric interpretation of the group M? Illustrate the action of the group elements in terms of transformations of a suitably chosen geometric object (analogous to the illustration of the dihedral group by its action on the stop sign in problem 1.3).

Bonus Problem

Problem 2.6. Systems of linear equations

A system of N linear equations of M variables $x_1, \ldots x_M$ comprises N equations of the form

$$\begin{aligned} b_1 &= a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1M}\, x_M \\ b_2 &= a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2M}\, x_M \\ \vdots & \quad \vdots \\ b_N &= a_{N1}\, x_1 + a_{N2}\, x_2 + \cdots + a_{NM}\, x_M \end{aligned}$$

where $b_i, a_{ij} \in \mathbb{R}$ for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$.

a) Demonstrate that the linear equations $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$ form a vector space when one adopts the operations

$$\begin{aligned} \forall \quad \mathbf{p} &= \big[p_0 = p_1\, x_1 + p_2\, x_2 + \cdots + p_M\, x_M\big] \in \mathbb{L}_M\,, \\ \mathbf{q} &= \big[q_0 = q_1\, x_1 + q_2\, x_2 + \cdots + q_M\, x_M\big] \in \mathbb{L}_M\,, \\ & c \in \mathbb{R}\ : \end{aligned}$$

$$\begin{aligned} \mathbf{p} + \mathbf{q} &= \big[p_0 + q_0 = (p_1 + q_1)\, x_1 + (p_2 + q_2)\, x_2 + \cdots + (p_M + q_M)\, x_M\big] \\ c \cdot \mathbf{p} &= \big[c\, p_0 = c\, p_1\, x_1 + c\, p_2\, x_2 + \cdots + c\, p_M\, x_M\big]\,. \end{aligned}$$

How do these operations relate to the operations performed in Gauss elimination to solve the system of linear equations?

b) The system of linear equations can also be stated in the following form

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M$$

$$\Leftrightarrow \qquad \mathbf{b} = x_1\, \mathbf{a}_1 + x_2\, \mathbf{a}_2 + \cdots + x_M\, \mathbf{a}_M$$

where $\mathbf{b}$ is expressed as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_M$ by means of the numbers $x_1, \ldots, x_M$. What do the conditions on linear independence and representation of vectors by means of a basis tell about the existence and uniqueness of the solutions of a system of linear equations.