# Homework Exercises 1

Chapter 1 and 2.1–2.3 of my lecture notes provide the background to solve the following exercises. Your solution to the problems 1.1–1.3 should uploaded

to your cloud-folder

as a PDF-file

by Saturday, Nov 7

Access information to the cloud-folders will be communicated by email next week. In the week starting Nov 9 we will organize seminar groups where the solutions will be discussed.

In the self-test exercises you can check you background and understanding of 1.4 Intervals and 1.5 Trigonometric functions. You will need this background to further follow the lecture, and to pass the exam. I will publish the solutions at the end of next week, and I will henceforth assume that you can deal with intervals, symmetries of trigonometric relations, and know about their common function values.

Bonus problems are illuminating fun problems for those participants who might be bored otherwise. Everybody else can safely ignore them.

Problems marked by might take some extra effort to solve.

## Problems

### Problem 1.1. Earth orbit around the sun

a) Light travels with a speed of $c \approx 3 \times 10^8\,\mathrm{m/s}$. It takes 8 minutes and 19 seconds to travel from Sun to Earth. What is the distance $D$ of Earth and Sun in meters? Why is it admissible for this estimate to assume that the light takes 500 seconds for the trip?

b) The period, $T$, of the trajectory of the Earth around the Sun depends on $D$, on the mass $M = 2 \times 10^{30}\,\mathrm{kg}$ of the sun, and on the gravitational constant $G = 6.7 \times 10^{-11}\,\mathrm{m^3/kg\,s^2}$. Estimate, based on this information, how long it takes for the Earth to travel once around the sun.
**Hint:** Which combination of $D$, $M$, and $G$ has a unit of seconds?

c) Express your estimate in terms of years. The estimate of (b) is of order one, but still off by a considerable factor. Do you recognize the numerical value of this factor?

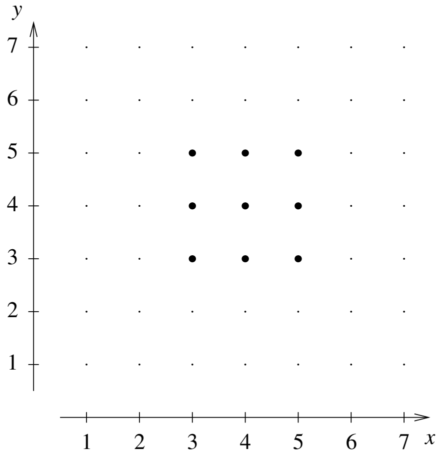
**Problem 1.2. Challenges in drawing circles**

In the realm of real numbers the set $K = \{(x, y) \in \mathbb{R} \mid (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$ describes a filled circle. Now we consider the set $N = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq x, y \leq 7\}$. Which of the following subsets of $N$ can be specified as a "circle" in the form

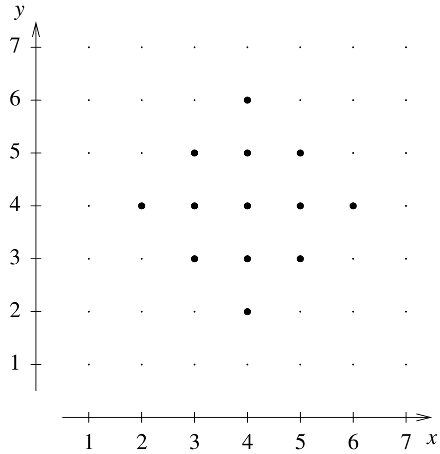
$$K_r = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x - 4)^2 + (y - 4)^2 \leq r^2\}$$

Which values of $r$ are admissible if this is possible?

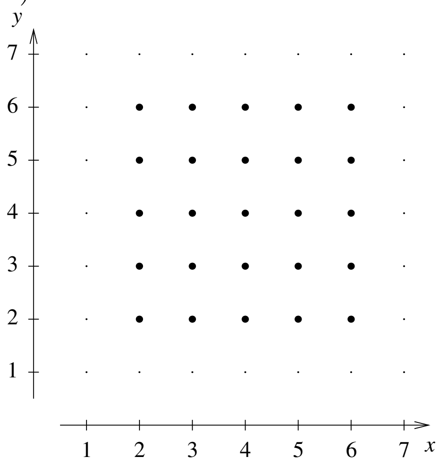
a)



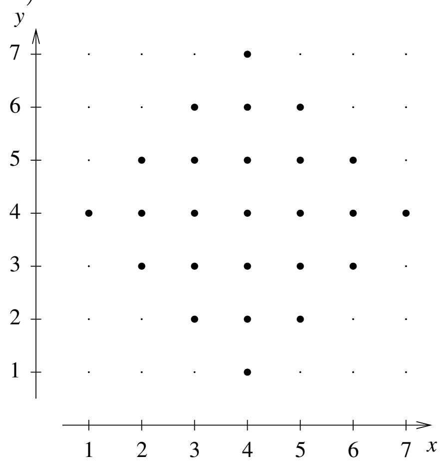

b)




c)




d)



**Problem 1.3. Dihedral group $D_8$**

In the figure below we demonstrate the action of the Dihedral group $D_8$ by its action on a stop sign. To this end we show the result of applying any of its elements to the configuration in the top left.


a) How many mirror axes and how many distinct rotation angles will transform the octagon in itself?

b) Find a rotation $d$ and a reflection $s$ that allow you to generate each element of $D_8$ by repeated action of $d$ and $s$. Specify how they should be applied to generate the 16 elements of $D_8$.
**Hint:** Presently the pictures are arranged in such a way that moving to the right always amounts to the same rotation. Rearrange the pictures in the lower row in such a way that the images in the bottom are obtained by reflection with respect to a horizontal line, as for the leftmost signs. How are neighboring signs in the lower row related in that case?

Bonus. Note: Some choices of rotations and reflections do not work. Can you find an example?

c) How many times do you have to apply $d$ to generate the neutral element of $D_8$? How many times do you have to apply $s$?

d) How can you represent $sds$ as composition of only $d$?

e) What is still to be done to proof that $D_8$ is a group?

## Self Test

**Problem 1.4. Intervals of real numbers**

Describe the following subsets of $\mathbb{R}$ as unions of disjoint intervals

a) $[-1,4]\backslash[1,2[$ b) $[2,4)\cup([3,10]\backslash(]3,4[\cup[6,7]))$

**Problem 1.5. Properties of right-angled triangles**

a) Fill in the gaps for the values of the angle $\theta$ in radians, and employ the symmetry of the trigonometric sine and cosine functions to determine the values in the right columns

| $\theta$ ° | rad | $\sin\theta$ | $\cos\theta$ |
|---|---|---|---|
| 0 | -------- | 0 | -------- |
| 30 | -------- | $\frac{1}{2}$ | -------- |
| 45 | -------- | $\frac{\sqrt{2}}{2}$ | -------- |
| 60 | -------- | $\frac{\sqrt{3}}{2}$ | -------- |
| 90 | -------- | 1 | -------- |
| 120 | -------- | -------- | -------- |
| 135 | -------- | -------- | -------- |
| 150 | -------- | -------- | -------- |
| 180 | -------- | -------- | -------- |

b) Consider a right triangle where one of the angles is $\theta$. How are the length of its sides related to $\sin\theta$ and $\cos\theta$? Check that the Theorem of Pythagoras holds! Do you see a systematics for the values provided for $\sin\theta$?

c) Use the symmetries of the trigonometric functions to determine the values provided for $\theta = \pi/4$.

Use the symmetries of the trigonometric functions and the trigonometric relation $\sin(2\theta) = 2\,\sin\theta\,\cos\theta$ to determine the values provided for $\theta = \pi/6$ and $\theta = \pi/3$.

The values for $\pi/10$, $\pi/8$, and $\pi/5$ can also be stated explicitly in elementary form. Determine the expressions for these values!

## Bonus Problem

### Problem 1.6. A group where the operation is a cross sum

We explore the set $M = \{1, 2, 4, 5, 7, 8\}$, and combine the elements by multiplication and subsequently taking the cross sum. For example $3 \circ 5 = 6$, because the product of 3 and 5 is $3 \cdot 5 = 15$, and the cross sum of 15 is $1 + 5 = 6$. For larger numbers we repeatedly take the sum of the digits until we arrive at a single-digit number. For instance, $7 \circ 8 = 2$, because $7 \cdot 8 = 56$ with cross sum $5 + 6 = 11$, and eventually we obtain $1 + 1 = 2$. We will show now that $(M, \circ)$ is a group, and discuss its relation to the dihedral group $D_3$.

a) Verify that the operation $\circ$ on $M$ is commutative.

b) Fill in the Caley table for the operation $\circ$ on $M$. How do you see in this table that the operation is commutative?

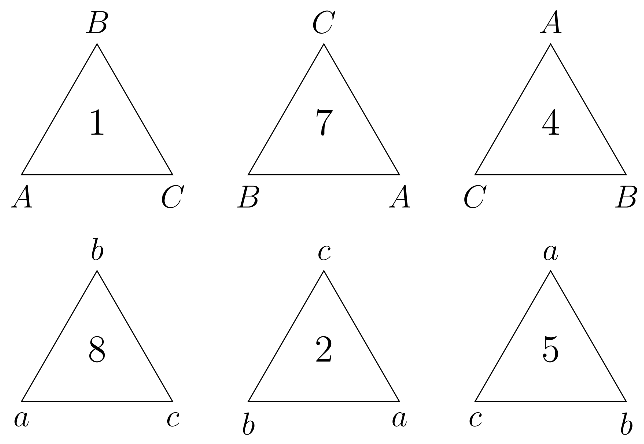
| $\circ$ | 1 | 2 | 4 | 5 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 7 | | | | | | |
| 8 | | | | | | |

c) Verify that the set $M$ is closed with respect to the operation $\circ$. Besides by checking the Caley table, this can also be shown by a direct calculation. How?
**Hint:** Observe the divisibility rule for 3 and 9.

d) What is the neutral element of this group?

e) What are the neutral elements for the other elements of $M$?

f) We arrange the elements graphically as follows:




Demonstrate that the element 7 amounts to a 120° rotation of the triangles. Henceforth, we denote it as $d$.
Verify that the element 8 swaps between small and capital letters at the vertices. Henceforth, we denote it as $s$.

g) How can the elements of the group be expressed in terms of $s$ and $d$?

h) Use the commutatively of the operation and the representation of the group elements obtained in (g) to proof associativity.