Blatt 2. Probability Distributions

The sign indicates exercises whose solution might require additional study and some effort.

Problems

2.1. Random walks (discrete in space and time)

We gamble on head and tail for coin flipping and certain sets of outcomes of dice. Proof the following statements.

a) When the coin tossing is fair, the probability to encounter h heads in N tosses is

$$P(h, N) = \frac{N!}{h!\,(N-h)!\,2^N}\,. \tag{2.1a}$$

When it is head we win one Euro, for tail we loose one Euro. How much money do we expect to win after N tosses, and what is the variance?

b) When heads are encountered with probability p in every single flip, the probability to encounter h heads in N tosses is

$$P(h, N) = \binom{N}{h}\, p^h\, (1-p)^{N-h}\,. \tag{2.1b}$$

How much money do we expect to win after N tosses, and what is the variance?

c) Now we play with a loaded die where we distinguish the events $\Omega = \{1 : (1 \text{ or } 6), 2 : (2 \text{ or } 5), 3 : (3 \text{ or } 4)\}$. The event i with $i =\in \{1, 2, 3\}$ entails winning w_i Euro (negative values imply that one looses). Show that the probabily to encounter n_i encounters of event $i \in \Omega$ in $N = \sum_{i\in\Omega} n_i$ throws of the die is

$$P(n_1, n_2, N) = \binom{N}{n_1\, n_2\, n_3}\, p_1^{n_1}\, p_2^{n_2}\, p_3^{n_3}\,, \tag{2.1c}$$

where p_i denotes the probability to enounter event $i \in \Omega$, and $n_3 = N - n_1 - n_2$.
What is the expected gain after N throws?

d) When $w_2 = w_3$ it is sufficient to distinguish the events $\{1\}$ and $\{2, 3\}$.
Show that the the expected gain after N throws can then be obtained from Equation (2.1b) with a substantially simpler calculation.
In order to understand why this must be the case: Recover distribution Equation (2.1b) by summation of Equation (2.1c) over all admissible values of n_2 and subsequently identifying n_1 with h. What is the relation between p and $\{p_1, p_2, p_3\}$?

2.2. Generating function for random walks

The calculations of the moments in exercise 3 simplify a lot when one considers generating function for the gain. Let us gain the amount x and y upon encountering head or tail in coin tossing. Hence, after encountering h heads in N tosses we gain $g_N = x\,h + y\,(N-h)$. Therefore, the n^{th} moment of the gain distribution amounts to

$$\begin{aligned}
\langle g_N^n \rangle &= \langle (x\,h + y\,(N-h))^n \rangle \\
&= \sum_{h=0}^{N} (x\,h + y\,(N-h))^n \binom{N}{h} p^h\,(1-p)^{N-h} \\
&= \frac{\mathrm{d}^n}{\mathrm{d}t^n} \sum_{h=0}^{N} \mathrm{e}^{(x\,h+y\,(N-h))\,t} \binom{N}{h} p^h\,(1-p)^{N-h} = \frac{\mathrm{d}^n}{\mathrm{d}t^n} G_N(t) \\
\text{with}\quad G_N(t) &:= \left\langle \mathrm{e}^{g_N t} \right\rangle = \left(p\,\mathrm{e}^{x\,t} + (1-p)\,\mathrm{e}^{y\,t}\right)^N
\end{aligned} \tag{2.2}$$

a) Use Equation (2.2) to verify that the binomial distribution, Equation (2.1b), is normalized and to calculate the first three moments of the expected gain.

b) Determine a generating function for the gain in exercise 2.1(c), and calculate the expectation and the variance of the expected gain.

2.3. The bank wins – almost certainly

We gamble on head and tail for coin flipping: We start with X Euros, winning a Euro for head and loosing a Euro for tail. Upon encountering zero Euro we go bancrupt and will remain bancrupt till the end of the game. Head and tail arise with the same probability $p = 1/2$.

a) Write a Sage or Python script to study how the probability $P_X(n,N)$ of starting with X Euros and owning n Euros after N rounds of the game evolves in time.

b) We first consider the initial phase of the game, where $N < X$. Show that the probability of starting with X Euros and owning n Euros after N rounds of the game amounts to

$$P_X^{(i)}(n,N) = \begin{cases} \dfrac{2^{-N}\,N!}{\left(\frac{N+n-X}{2}\right)!\,\left(\frac{N-n+X}{2}\right)!} & \text{for}\quad X-N \le n \le X+N, \\ 0 & \text{else} \end{cases} \tag{2.3a}$$

as long as $N < X$.

c) There is no issue of bancrupcy in the result of b) such that the same expression would apply if we were allowed to play with negative credit $X < 0$. Show that for all N the probability of starting with X Euros and owning $n > 0$ Euros after n rounds of the game amounts to

$$P_X(n,N) = P_X^{(i)}(n,N) - P_{-X}^{(i)}(n,N) \tag{2.3b}$$

Here $P_{-X}^{(i)}(n,N)$ is the credit of a gambler that starts with credit $-X$, and does not encounter special treatment for zero credit.

d) Discuss the normalization of the probablity distribution.

2.4. The characteristic function of the Lorentz-Cauchy distribution

The Lorentz-Cauchy distribution is defined as

$$p_{LC}(x) = \frac{1}{\pi} \frac{\Gamma}{(x-m)^2 + \Gamma^2} \qquad \text{with parameters } m, \Gamma \in \mathbb{R} \tag{2.4a}$$

a) Verify that the distribution is normalized.

b) Determine the expectation value of the distribution.
What about the variance?

c) Show that the characteristic function of $p_{LC}(x)$ is

$$\chi_{LC}(t) = \left\langle \mathrm{e}^{\mathrm{i}\, t x} \right\rangle = \mathrm{e}^{\mathrm{i}\, m\, t - \Gamma\, |t|} \tag{2.4b}$$

What does this tell about the normalization, expectation and variance?

⋆d) Consider the distribution

$$p_4(x) = \frac{4}{\pi} \frac{\Gamma^3}{(x-m)^4 + 4\,\Gamma^4} \qquad \text{with parameters } m, \Gamma \in \mathbb{R} \tag{2.4c}$$

Determine the characteristic function.
Show that it provides m and $2\Gamma^2$ for the expectation and the variance, respectively.
What happens for higher moments?

2.5. Uncorrelated vs. independent variables

We consider the following probability distribution

$$p(x,y) = c\,(x^2 + y^2) \qquad \text{for} \qquad (x,y) \in [-1,1] \times [-1,1]\,.$$

a) Determine the normalization constant.

b) Determine the marginal probabilities $p_1(x) = \int \mathrm{d}y\, p(x,y)$ and $p_2(y) = \int \mathrm{d}x\, p(x,y)$.

c) Determine the conditional probabilities $p(x|y)$ and $p(y|x)$.

d) Determine the expectation values $\langle x \rangle$, $\langle y \rangle$, and $\langle x\, y \rangle$.

Bonus-Aufgabe

2.6. Bertrand's paradox

A line is dropped randomly on a circle. The intersection will be a chord. What is the probability that the length of the chord is larger than that of a side of the inscribed equilateral triangle?
Bertrand's paradox states that the resulting probabiliy depends on the rule how the lines are selected. To understand this observation we consider intersections with the unit circle at the origin.

a) Specify the line by two points: The first point is $P_1 = (0, 1)$ on the circle and a second point $P_2 = (x, y)$ is selected at random in $\mathbb{R}^2$.
Determine the probability distribution for the length of the chords, and its expectation value.
Hint: What is the probability density to find the second point in a direction ϕ with respect to the x-axis, when looking from P_1?

b) Pick as first point $P_1 = (0, 1)$ on the circle, as before. However, now the second point $P_2 = (x, y)$ is selected at random in $[-L, L] \times [-L, L]$ for some $L \in \mathbb{R}^+$.
Determine the probability distribution for the length of the chords and its expectation value. How does the result depend on L?

c) Employ rotational symmetry and only consider horizontal lines (x, y_h) with $x \in \mathbb{R}$ and fixed y_h. Let y_h be uniformly distributed in $[-1, 1]$.
Determine the probability distribution for the length of the chords and its expectation value.

d) Write a Python program to support your findings and explore other settings.