Exam — Self Test

Instructions for working on this exam

1. In the final exam there will be 120 min to execute the exam. In total there are 60 points + about 25 bonus points. You pass the exam with 25 points. For the best grade, 1.0, you need 45 points. The points for the different sub-tasks are provided on the next page. If you want to go through the full exam in two hours, there will be about 2 min/point.

2. In the present exam there are three exercises that address skills that we need to solve physical problems, and subsequently you will discuss in some depth two physical problems.

3. The best strategy to attack the exam is to strive to collect as many points as possible. Do not try to solve the full exam. Work on things that you can solve — ignore anything where you get stuck.

4. Each exercise should be solved on a separate set of sheets. Put your name and matrikel number on the top left of every sheet.

5. Bonus problems are marked by (*). Often they involve a tricky argument that might not be immediately obvious. I recommend that you first work on the other exercises. Only attack the *-problems in the end when there is still time, or when you immediately see a fast and straightforward solution.

6. Explain carefully what you are doing. What do you assume? What do you intend to show? We will only give points when we understand what you intend to do.

7. You may use one sheet of A4 paper with notes and equations. Think carefully what is most useful for you to put on that paper.

name, given name

matrikel nummer

1 (a)	(b)	∗[c]	2 (a)	(b)	3 (a)	(b)	∗[c]	∗[d]	4 (a)	(b)
3	4	3	4	4	3	4	4	2	4	4

(c)	(d)	∗[e]	∗[f]	5 (a)	(b)	(c)	(d)	(e)	∗[f]	(g)
5	3	6	4	2	5	4	3	4	6	4

total points

grade

Problem 1. Forces, potentials, and line integrals

Consider the force $\vec{F} = (x + yz, y + xz, z + xy)$.

a) Determine $\nabla \times \vec{F}$.

b) Evaluate the line integral $\int_\gamma \mathrm{d}\vec{s} \cdot \vec{F}$ for the path $\gamma = (at, bt, ct)$ with some real constants a, b, c and $t \in [0, 1]$.

** c) Is the force $\vec{F}$ conservative? Substantiate your reply!
If no: Why not?
If yes: Determine the potential $\Phi(x, y, z)$ with $\vec{F} = -\nabla\Phi(x, y, z)$.

Problem 2. Equipotential lines and gradients

a) Determine the equipotential lines and the gradients of

$$\Phi(x, y) = \cosh(x - y)\,.$$

b) Sketch the equipotential lines of the potential, and mark the gradients by arrows.

Problem 3. Dynamics in phase space.

We consider the following equations of motion

$$\dot{x}(t) = \alpha\, x(t)\, \big(p(t) - 1\big)\,, \qquad \dot{p}(t) = \quad p(t)\, \big(1 - x(t)\big)$$

where α is a positive real number, and $x(t)$, $p(t)$ are real-valued funktions.

a) Determine the fixed points of the dynamics,
i.e. points (x_0, p_0) where $\big(\dot{x}(t),\, \dot{p}(t)\big)_{(\dot{x}(t),\, \dot{p}(t))=(x_0,p_0)} = 0$.

b) Show that

$$I = x(t) + \alpha\, p(t) - \ln\big(x(t)\, p^\alpha(t)\big)$$

is a constant of motion of the dynamics, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} I = 0$.

** c) For small ϵ one has $1 + \epsilon - \ln(1 + \epsilon) \simeq 1 + \epsilon^2/2$. Use this information to sketch the phase-space trajectories of the dynamics.

** d) Provide and argument why the functions $x(t)$ and $p(t)$ can not be considered as the position and the momentum of a particle.

Problem 4. A trip on the rollercoaster.

We parameterize the height H of a rollercoaster track by the contour length ℓ of the track. In some rough approximation this will lead to a biquadratic funktion

$$H(\ell) = a\,(\ell^2 - L^2)^2$$

where $H(\ell)$ is the height over the lowermost point of the track.

a) Sketch $H(\ell)/(aL^4)$ as function of ℓ/L.

b) What are the units of a and L?
What is the height of the looping?
Which distance does a car travel from the lowest point of the track, once through the looping, till it arrives at the following minimum?

Rollercoaster Tornado im Avonturenpark Hellendoorn, Niederlande
[public domain, wikimedia.commons]

c) To start with the consider the motion of a single car of mass m on the track. Determine its kinetic and potential energy as function of $\ell(t)$.
Verify that its equation of motion takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\ell(t)}{L} = \omega^2\,\frac{\ell(t)}{L}\left(\frac{\ell(t)}{L} - 1\right)\left(\frac{\ell(t)}{L} + 1\right)$$

How is ω related to a, L, and the gravitational acceleration g?

d) Sketch the solutions in phase space.

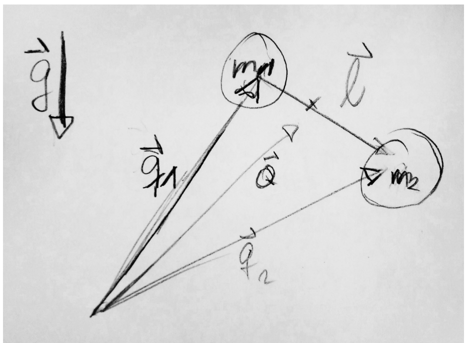
** e) Consider now two cars of the same mass m. We model them as two point masses that move along the track with a fixed distance $2w$. Let $\ell(t)$ denote the position of the center of mass of the two cars. Show that the kinetic energy and the potential energy will then take the form

$$T = m\,\dot{\ell}^2(t)$$
$$V = 2\,m\,g\,a\,\left[(\ell^2 - A^2)^2 + B^2\,\ell^2\right]$$

How do A and B depend on w and L?

** f) Discuss also the case of three cars of mass m and distance $2w$.

Problem 5. Flight of a dumbbell.



We explore the flight of a dumbbell under the influence of gravity $\vec{g}$ in our three-dimensional space. The dumbbell is idealized as two particles of masses m_1 and m_2. They positions $\vec{q}_1(t)$ and $\vec{q}_2(t)$ will be kept at a distance ℓ by a bar of negligible mass. We denote the center of mass of the dumbbell as $\vec{Q}$ and the relative coordinate as $\vec{\ell} = \vec{q}_2 - \vec{q}_1$.

a) We express the ralation between $(\vec{Q}, \vec{\ell})$ and the positions $\vec{q}_i$, $i \in \{1, 2\}$ as $\vec{q}_i = \vec{Q} + \alpha_i\, \vec{\ell}$. Determine the real numbers α_i, $i \in \{1, 2\}$.

b) Show that the kinetic energy and the potential energy of the dumbbell have the form

$$T = \frac{M}{2}\,\dot{\vec{Q}}^2 + \frac{\mu}{2}\,\dot{\vec{\ell}}^2 ,$$
$$V = -M\vec{g}\cdot\vec{Q} + \Phi(\ell)$$

where $\Phi(\ell)$ is a potential that will generate the force fixing the dislance of the masses to the value ℓ. How do M and μ depend on m_1 and m_2?

c) Show that

$$\ddot{\vec{Q}} = \vec{g}$$

How does the trajectory of the center of mass of the dumbbell look like when the dumbbell is thrown at time t_0 from a position $\vec{Q}_0$ with a velocity $\vec{V}_0$?

d) Show that

$$\mu\,\ddot{\vec{\ell}} = -\hat{\ell}\cdot\nabla\Phi(\ell) \quad \text{with} \quad \hat{\ell} = \frac{\vec{\ell}}{\ell}.$$

e) Show that the energy $E = \mu\dot{\vec{\ell}}^2/2 + \Phi(\ell)$ and the angular momentum $\vec{L} = \mu\vec{\ell}\times\dot{\vec{\ell}}$ are constants of the motion of the dumbbell.

** f) Write the rotational motion of the dumbbell as $\dot{\vec{\ell}} = \Omega\times\vec{\ell}$ with a constant vector $\vec{\Omega}$. Verify by explicit calculations that this ansatz fulfills the requirements on the conservation of the distance between the masses, the energy and the angular momentum.

g) Provide the position of the masses $\vec{q}_1(t)$ and $\vec{q}_2(t)$ for the initial conditions provided in (c), some fixed Ω, and $\vec{\ell}_0$.