Homework Exercises 13

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Jan 27, at 11:00,
or in my mail box **in the Linnéstr**, by January, Dec 27, 13:00.
Please submit only Problem 13.4 when you participate in the seminar.
The warm-up problems will be addressed in the question session on Friday.

Warm-up

Problem 13.1. Effective potentials and phase-space portraits.

We consider ODEs of the form

$$\ddot{x}(t) = -\frac{\mathrm{d}}{\mathrm{d}x} V_{\text{eff}}(x)$$

Sketch the solutions for trajectories in the following potentials in the phase space $(x, \dot{x})$.

a) $V_{\text{eff}} = x \, \sin x$ b) $V_{\text{eff}} = x \, \cos x$
c) $V_{\text{eff}} = x - \sin x$ d) $V_{\text{eff}} = x - \cos x$
e) $V_{\text{eff}} = \mathrm{e}^{x} \, \sin x$ f) $V_{\text{eff}} = \mathrm{e}^{-x} \, \sin x$

Homework Problems

Problem 13.2. The steel-can pendulum

We consider a pendulum that is built from a steel can with two heavy magnets of identical mass M attached inside. When the can is released it starts oscillating. When pushed, it wobbles along a straight path. The angele between the two magnets is denoted as α. We describe the configuration of the can by the angle θ that follows how it is rolling, and we choose the origin of θ such that the magnets are positioned at $\pm\alpha/2$.

a) Assume that the mass of the can may be neglected as compared to that of the magnets. Determine the kinetic energy, T, and the potential energy, V, of the can. Use the Lagrange formalism to determine the equation of motion for θ.

b) For $\alpha = 0$ you should find the following equation of motion

$$0 = 2\ddot{\theta}\,\big(1 - \cos\theta\big) + \dot{\theta}^2 \sin\theta + \omega^2\, \sin\theta \tag{13.1}$$

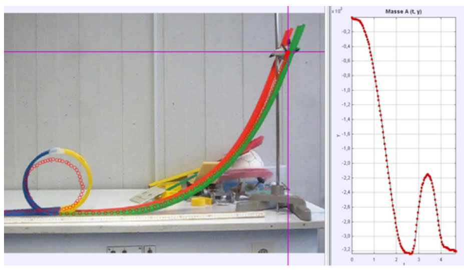
How does ω depend on the radius R of the can, the mass M, and the gravitational acceleration g?

c) Discuss the case $\alpha = \pi$. How does the can move in this case? Sketch the trajectories in phase space.

d) Determine the equilibrium position of the can for $\alpha \neq 0$, and provide a physical argument why they are stable of unstable. Sketch the according motion in phase space.

Problem 13.3. Ball and Darda tracks

Building ball tracks and Darda tracks is a fascinating challenge for small and large children. I stared trails from my wardrobe to build tracks with several loopings. Pupils of the Otto-von-Taube-Gymnasiums took videos of cars running in Darda tracks, and evaluate them by tracking the cars.



School project of a 9. Klasse at the Otto-von-Taube-Gymnasium, Gauting

Mathematically the motion along the trail is best studied by parametrizing the position of the car via the length ℓ of the path along the trail. The speed of the car is then $\dot{\ell}$; the height of the trail will be denoted as $H(\ell)$. The mass of the car is denoted as m.

a) Determine the Lagrange function and the resulting equations of motion for the position $\ell(t)$ of the car.

b) We first consider a piece of the trail with a constant angle ϕ with respect to the horizontal, $H(\ell) = H_0 - \ell \sin\phi$. The car starts at height H_0 with velocity zero. Solve the equation of motion.

Bonus: What is the maximum speed of the car when there is a constant friction force $-\gamma\dot{\ell}$? How does the solution of the equations of motion look like in that case?

Hint: The equation of motion is a linear differential equation with constant coefficients.

c) After a speedy trip the car is oscillating in the final part of the trail where $H(\ell) = a\,(\ell - L)^2/2$. Here, a and L are positive constants with suitable units. What are the units of the constants?

   Solve the equation of motion for the case where the car goes through the position $\ell = L$ with speed v_0.

   Bonus: How does the solution look like when there is a constant friction force $-\gamma\dot{\ell}$?

d) The car is driving through a looping with radius R. In the looping we have $\ell = R\,\theta$ where θ indicates how far the car has moved with respect from the entrance of the looping right at its bottom. Determine the equation of motion for this piece of the trip and sketch the solutions in phase space.

   Bonus: Some trajectories are not admissible because the car will then fall off the track! What is the minimum speed of the car such that it does not fall? Which part of the phase space is then physically not admissible?

Problem 13.4. Evolution of a particle in a Mexican-hat potential

We explore the motion of a particle of mass m in a rotation-symmetric potential

$$\Phi(r) = \frac{m\,A}{4}\, r^2\,(r^2 - 2\,r_0^2)$$

The particle evolves in a plane where its position is specified by the polar coordinates (r, θ).

a) Sketch the potential. Where are its maxima and minima?

b) Determine the Lagrange function for this problem, and determine the equations of motion for $\theta(t)$ and $r(t)$.

   Bonus. The angular momentum and the energy of the particle are conserved. How do you see this without calculation based on the Lagrange function?

c) Determine a frequency ω, a length scale ℓ and a constant K, such that

$$\frac{\mathrm{d}^2\hat{r}}{\mathrm{d}(\omega t)^2} = \hat{r} - \hat{r}^3 + \frac{K}{\hat{r}^3} \qquad \text{with} \quad \hat{r}(t) = \frac{r(t)}{\ell}\,.$$

In the following we discuss the dimensionless equations, where we absorb ω into the time scale and drop the hat to avoid clutter in the equations.

d) Multiply the equation of motion by $\dot{r}$, and rewrite it in the form

$$E = \frac{\dot{\hat{r}}^2}{2\omega^2} + V_{\text{eff}}(\hat{r}) \qquad \text{with} \quad V_{\text{eff}}(\hat{r}) = \frac{\hat{r}^4}{4} - \frac{\hat{r}^2}{2} + \frac{K}{2\,\hat{r}^2}\,.$$

e) Sketch the effective potential $V_{\text{eff}}(\hat{r})$ and the phase portrait of the motion for $K > 0$.

Bonus. Why is it necessary to give a separate discussion of $K = 0$?

Bonus Problem

Problem 13.5. Solving the EOM of the steel-can pendulum for $\alpha = 0$

We will determine the evolution $\theta(t)$ for the steel-can pendulum where the magnets are attached to the same position at the circumference of the can.

a) Show that Equation (13.1) simplifies considerably upon introducing the new variable $\xi = \cos(\theta/2)$,

$$\ddot{\xi} = \frac{\omega^2}{4}\,\xi \tag{13.2}$$

Instructions: 1. Observe that $1-\cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.
2. Determine $\dot{\xi}$ and express $\dot{\theta}$ in terms of $\dot{\xi}$ and θ.
3. Determine $\ddot{\xi}$ and use this to discuss $2\,\sin(\theta/2)\,\ddot{\xi}$. Eliminate $\ddot{\theta}$ by the equation of motion. Subsequently, eliminate $\dot{\theta}$ by the relation obtained in 2.

b) Determine the evolution of the can when it is released with zero velocity.

c) Determine the evolution of the can when it is set into motion by a kick starting from the equilibrium position (i.e., the equilibrium fixed point).

d) Determine the general solution for $\theta(t)$ based on the solution for $\xi(t)$ obtained from Equation (13.2). What should be taken into account when $\sin(\theta/2) = 0$ such that one obtains a physically sound solution?