Homework Exercises 12

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Jan 20, at 11:00,
or in my mail box **in the Linnéstr**, by January, Dec 20, 13:00.
Please submit only Problem 12.3 when you participate in the seminar.

Homework Problems

Problem 12.1. The driven pendulum

We consider a (mathematical) pendulum of mass m with a pendulum arm of length ℓ. The deflection of the arm with respect to its rest position is denoted as $\theta(t)$. The fulcrum of the pendulum is moving vertically, residing at the position $z(t)$ at time t.

a) Sketch the setup of for this problem, and mark the relevant physical parameters and coordinates.

b) Determine the kinetic energy T and the potential energy V of the pendulum mass.

c) We assume that the mass of the pendulum arm does not play a significant role for the motion. Use the Lagrange formalism to determine the equation of motion for θ. Can you imagine why $\ddot{\theta}(t) = 0$ for $\ddot{z} = -g$?

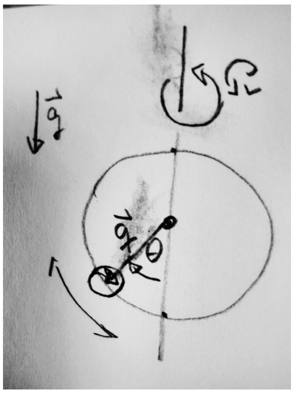
d) Discuss the stability of the fixed points of the motion for $\ddot{z} > -g$ and $\ddot{z} < -g$. Sketch corresponding phase-space plots for constant $\ddot{z}$.

Bonus Consider now the T-periodic motion with

$$z(t) = \begin{cases} a\,t\,(t - T/2) & \text{for} \quad 0 < t < T/2 \\ a\,(T-t)\,(t - T/2) & \text{for} \quad T/2 < t < T \end{cases}$$

What does this imply for $\ddot{z}$? How would the trajectories close to $\theta = 0$ and $\theta = \pi$ look like for $a > g$?

Problem 12.2. Phase-space analysis for a pearl on a rotating ring



We consider a pearl of mass m that can freely move on a ring that stands vertically in the gravitational field, and spins with angular velocity Ω around its vertical symmetry axis. Since the position of the pearl is constrained to lie on the surface of a sphere, we adopt spherical coordinates and describes its position as

$$\vec{q}(t) = \ell\, \hat{r}(\theta(t), \Omega t)$$

where θ is the deflection of the pearl from its rest position of the lowermost point of the ring, and $\phi = \Omega t$ the orientation of the ring in the horizontal plane.

a) Determine the components of $\hat{r}(\theta, \phi)$ in Cartesian coordinates where the z-axis is aligned antiparallel to $\vec{g}$. Verify then by explicit calculation that $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ with

$$\hat{\theta} = \frac{\partial \hat{r}}{\partial \theta} \qquad \text{and} \qquad \hat{\phi} = \hat{r} \times \hat{\theta}$$

form an orthonormal basis. How is $\hat{\phi}$ related to $\partial \hat{r}/\partial \phi$?

b) Evaluate $\dot{\vec{q}}(t) = \ell\, \dot{\hat{r}}(\theta(t), \Omega t)$.

c) Determine the kinetic energy T and the potential energy V of the pearl.

d) Use the Lagrange formalism to determine the equation of motion of the pearl. How is it related to the motion of the rotational governor studied in Problem 11.2?

e) Determine the fixed points for the motion of the pearl, and discuss their stability as function of Ω.

f) Sketch phase-space plots for the motion of the pearl.

Problem 12.3. The kitchen pendulum

We consider a pendulum that is built from two straws (length L_1 and L_2), two corks (masses m_1 and m_2), a paper clip, and some Scotch tape (see picture to the right). It is suspended from a shashlik skewer, and its motion is stabilized by means of the spring taken from a discharged ball-pen. Hence, the arms move vertically to the skewer. We denote the angle between the arms as α, and the angle of the right arm with respect to the horizontal as $\theta(t)$.

a) Determine the kinetic energy, T, and the potential energy, V, of the pendulum. Argue that T and V can only depend on θ and $\dot{\theta}$, and determine the resulting Lagrangian $\mathcal{L}(\theta, \dot{\theta})$.

b) Determine the EOM of the pendulum.

c) Find the rest positions of the pendulum, and discuss the motion for small deviations from the rest positions. Sketch the according motion in phase space.

d) The EOM becomes considerably more transparent when one selects the center of mass of the corks as reference point. Show that the center of mass lies directly below the fulcrum when the pendulum it at rest.

e) Let ℓ be the distance of the center of mass from the fulcrum, and $\varphi(t)$ be the deflection of their connecting line from the vertical. Determine the Lagrangian $\mathcal{L}(\varphi, \dot{\varphi})$ and the resulting EOM for $\varphi(t)$. Do you see how the equations for $\ddot{\theta}(t)$ and $\ddot{\varphi}(t)$ are related?

Bonus Problem

Problem 12.4. Mechanical similarity

Two solutions of a differential equations are called *similar* when they can be transformed into one another by a rescaling of the time-, length- and mass-scales. We indicate the rescaled quantities by a prime, and denote the scale factors as τ, λ, and μ, respectively,

$$t' = \tau t\,, \qquad \vec{q}_i' = \lambda \vec{q}_i\,, \qquad m_i' = \mu m_i$$

a) We consider a system with kinetic energy $T = \frac{1}{2} \sum_i m_i \dot{\vec{q}}_i^2$, and consider a potential that admits the following scaling

$$V' = \mu^\alpha \, \lambda^\beta \, V$$

Show that the EOM are then invariant when one rescales time as

$$\tau = \mu^{(1-\alpha)/2} \, \lambda^{(2-\beta)/2}$$

b) Consider now two pendulums, $V = mgz$ with different masses and length of the pendulum arms. Which factors τ, λ, and μ relate their trajectories? How will the periods of the pendulums thus be related to the ratio of the mass and the length of the arms? Which scaling do you expect based on a dimensional analysis?

c) What do you find for the according discussion of the periods of a mass attached to a spring, $V = k\,|\vec{q}|^2/2$?

d) Discuss the period of the trajectories in the Kepler problem, $V = mMG/|\vec{q}|$. In this case the dimensional analysis is tricky because the masses of the sun and of the planet appear in the problem. What does the similarity analysis reveal about the relevance of the mass of the planet for Kepler's third law?