Homework Exercises 11

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Jan 13, at 11:00,
or in my mail box **in the Linnéstr**, by January, Dec 13, 13:00.
Please submit only Problem 11.3a–c when you participate in the seminar.

Warm-up

Problem 11.1. Soda Bins

In 1984 RedBull introduced a slim 250 mL soda bin with a diameter of $D_{RB} = 53\,\text{mm}$ and a height of $H_{RB} = 135\,\text{mm}$. Up to that point the usual size was 300 mL with a bin diameter of $D_C = 67\,\text{mm}$ and a height of $H_C = 115\,\text{mm}$ (this is the size of classic beer- and Coke-bins).

a) A fist impression of the size is given by the visual cross section $D \times H$. Check it out: The cross section of RedBull and Coke bins is not too different — in spite of their substantial volume difference.

b) From an environmental point of view one would try to use as little aluminum for the bins as possible. This has indeed been achieved by reducing their wall thickness (cf. `https://www.youtube.com/watch?v=hUhisi2FBuw`). However: What would be the optimal form to maximize volume for a given surface area? Show in this respects that the surface area for fixed volume is proportional to

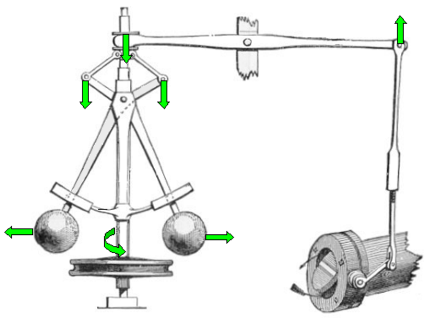
$$A = K\,\alpha^{-2/3}\,(1+\alpha) \qquad \text{where} \qquad \alpha = \frac{H}{R}$$

is the aspect ratio and K some constant. Determine the constant K, and the aspect ratio, α_{opt}, that minimizes the aluminum use.

Homework Problems

Problem 11.2. Error analysis: rotational governor

A rotational governor is used in steam engines to control the pressure. When a threshold value for the rotational frequency, Ω_c, is reached a valve is opened to reduce the pressure. For smaller frequencies, $\Omega < \Omega_c$, the two balls attached to the arms are hanging down. For high frequencies, $\Omega > \Omega_c$, they are pushed outwards, and steam is released. Due to the symmetry both balls have the same kinetic and potential energy. Hence, we consider only one ball.



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We describe its position by the the vertical position z, the distance from the rotation axis r, and the angle ϕ, of the position of the ball in the horizontal plain (cylindrical coordinates). The length of the arms is denoted as L, the mass of the balls as M, and the angle between the arm and the vertical rotation axis as θ. The governor is rotating with frequency Ω.

a) Sketch the setup, indicating the relevant parameters and coordinates. Indicate in particular also the unit vectors of the orthonormal basis for the coordinates (z, r, ϕ). In this basis the position of the ball should take the form:

$$\vec{q}(t) = -L\,\cos\theta(t)\;\hat{z} + L\,\sin\theta(t)\;\hat{r}(\phi(t)) \qquad \text{mit} \quad \phi(t) = \Omega t\,.$$

b) Determine the kinetic energy E_{kin} of the ball, and its potential energy E_{pot} in the gravitational field.

c) Discuss whether the sum of kinetic and potential energy is conserved for the motion. Determine to this end the time derivative $E_{\text{kin}} + E_{\text{pot}}$, and show that the time derivative vanishes if and only $\dot{\theta} = 0$ or

$$\ddot{\theta}(t) = -\omega^2 \sin\theta(t)\;\left[1 + \frac{\Omega^2}{\omega^2}\cos\theta(t)\right] \qquad \text{mit} \quad \omega^2 = \frac{g}{L}\,.$$

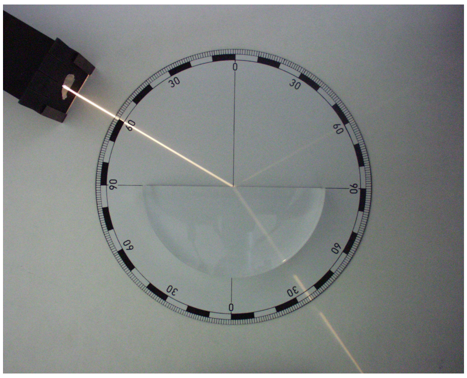
d) The rotational governor will always have some small friction $-\gamma\dot{\theta}$. Analyze therefore the equation of motion

$$\ddot{\theta}(t) = -\omega^2 \sin\theta(t)\;\left[1 + \frac{\Omega^2}{\omega^2}\cos\theta(t)\right] - \gamma\dot{\theta}$$

What happens for slow rotation $\Omega^2 \ll \omega^2$? What changes when $\Omega^2 \gtrsim \omega^2$? Does this result comply with your *physical* intuition? If not: Why not? What could have gone wrong?

Problem 11.3. Fermat's principle

Fermat's principle states that a light beam propagates along a path minimizing the flight time. When passing from air into glass it changes direction according to Sellius' refraction law.



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Here, we consider a setting where the beam starts in air at the position, $(x, y) = (0, 0)$, to the top left in the figure, with coordinates where $\hat{x}$ points downwards and $\hat{y}$ to the right. The path of the light is described by a function $y(x)$. We require that beam passes from air into the glass at the position (a, u) such that it will eventually proceed through the prescribed position (b, w) in the glas. The speed of light in air and in glass will be denoted as c_L and c_G, respectively.

a) Show that the time of flight T for a (hypothetic) trajectory $y(x)$ with derivative $y'(x)$ can be determined as follows

$$T = c_L^{-1} \int_0^a \mathrm{d}x \, \sqrt{1 + (y'(x))^2} + c_G^{-1} \int_a^b \mathrm{d}x \, \sqrt{1 + (y'(x))^2} \, .$$

b) Determine δT for a variation $y(x) + \varepsilon \, \delta y(x)$ of the trajectory. We describe the glass surface by a function $s(x)$, but we do not know at which position along the surface the beam passes from air into the glas. What does this imply for $\delta y(x)|_{x=0}$, $\delta y(x)|_{x=a}$ and $\delta y(x)|_{x=b}$? What does it imply for the boundary terms that arise from the integration by parts, when determining δT?

c) Show that the beam must go in a straight line in air and in glas. Show that this implies that

$$T(u) = \frac{1}{c_L} \sqrt{u^2 + a^2} + \frac{1}{c_G} \sqrt{(w - u)^2 + (b - a)^2} \, .$$

Derive Snellius' law from the conditin that $0 = \mathrm{d}T(u)/\mathrm{d}u$.

Bonus. Snellius' Law can also be directly obtained from Fermat's principle. How?

Problem 11.4. Shortest path on a sphere

We describe the position on the surface of a three-dimensional sphere by the angle θ with its "North pole", and the azimuthal angle ϕ in the horizontal plane. A trajectory on the sphere can then be specified as $\vec{q}(t) = \big(\theta(t), \phi(t)\big)$, or alternatively by $\theta(\phi)$ or $\phi(\theta)$. We will now derive conditions for a path of extremal length on the sphere.

a) Without restriction of generality we restrict our discussion to spheres with unit ratius. Why is this admissible?

b) Show that the length of the path from (θ_a, ϕ_a) to (θ_e, ϕ_e) amounts to

$$L = \int \mathrm{d}\ell = \int_{\phi_a}^{\phi_e} \mathrm{d}\phi \, \sqrt{\sin^2\theta(\phi) + \left(\frac{\mathrm{d}\theta(\phi)}{\mathrm{d}\phi}\right)^2}$$
$$= \int_{\theta_a}^{\theta_e} \mathrm{d}\theta \, \sqrt{1 + \sin^2\theta \, \left(\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}\right)^2}$$

Under which conditions do the expressions apply? Why and when do they provide the same length?

c) A necessary condition for the extremality of L is that the variation δL vanishes for the integrals that have been defined in (b). Introduce the variation $\theta(\phi) + \varepsilon\,\delta\theta(\phi)$ into the second representation of thelength, calculate δL, and determine the resulting differential equation for paths $\theta(\phi)$ of extremal length.

d) Repeat the same steps for the variation $\phi(\theta) + \delta\phi(\theta)$ and the other representation of the length. Determine the resulting differential equation for paths $\phi(\theta)$ of extremal length. Which derivation is simpler? How could you have seen this *before* performing the calculations?

e) The result of (d) can be integrated once. Show that this results in the following first order differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \frac{\cos\alpha}{\sin\theta} \, \left[\sin^2\alpha - \cos^2\theta\right]^{-1/2} .$$

where $\sin\alpha$ is an integration constant.

f) Verify that the following function is a solution of the differential equation

$$\sin\phi = -\frac{\cos\alpha \, \cos\theta}{\sin\alpha \, \sin\theta} .$$

There is no further integration constant in this solution! What does that imply?

g) Show that all the coordinates

$$\vec{q}(\theta) = (\sin\theta\,\cos\phi(\theta), \sin\theta\,\sin\phi(\theta), \cos\theta)$$

of the trajectory obtained in (f) are orthogonal to the vector $(0, \sin\alpha, \cos\alpha)$. What does this imply for the path?

Bonus Problem

Problem 11.5. Stability of soap films

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When a soap film is suspended between two rings, it takes a cylinder-symmetric shape of minimal surface area. We discuss here the form of the film for rings of radius R_0 and R_1 positioned at the height x_0 and x_1, respectively. At the Mathematikum in Gießen there is a nice demonstration experiment: x_0 is the surface height of soap solution in a vessel around the platform where the children are standing, and x_1 is the height of the ring pulled upwards by the children.

http://mathematikum.df-kunde.de/Wanderausstellung/index.php?m=2&la=de&id=314

a) Let $w(x)$ be the radius of the cylinder-symmetric soap films at the vertical position x. Sketch the setup and mark the relevant notations for the problem.

b) Show that the surface area A of the soap film takes the form to

$$A = \int_{x_0}^{x_1} \mathrm{d}x\, w(x)\, f(w'(x))\,,$$

Here, the factor $f(w'(x))$ takes into account that the area is larger when the derivative $w'(x) = \mathrm{d}x/\mathrm{d}x$ increases. Determine the function $f(w'(x))$ in this expression.

c) Show that A is extremal for shapes $w(x)$ that obey the differential equation

$$w''(x) = \frac{1 + (w'(x))^2}{w(x)} \,.$$

d) Determine the solutions of the differential equation.
Hint: Rewrite the equation into the form

$$\frac{w'(x)\, w''(x)}{1 + (w'(x))^2} = \frac{w'(x)}{w(x)} \,.$$

e) Look now for solutions with $-x_0 = x_1 = a$ and $R_0 = R_1 = R$, where w_0 is the radius at the thinnest point of the soap film. Show that the solution will then take the form

$$\frac{R}{a} = \frac{w_0}{a} \cosh \frac{a}{w_0} \,.$$

f) Sketch R/a as function of a/w_0. For given R and a you can then find w_0. For small separation of the rings you should find two solutions. What happens when one slowly rises the ring? Will an adult ever manage to pull up the ring to head height before the film ruptures?