Homework Exercises 10

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Jan 6, at 11:00,
or in my mail box **in the Linnéstr**, by January, Dec 6, 13:00.
Please submit only Problem 10.2 when you participate in the seminar.

Warm-up

For the holidays I recommend some fun literature to read and contemplate about

a) Rigid-body dynamics of a football
Peter J. Brancazio
`https://doi.org/10.1119/1.15123`

b) The right spin: how to fly a broken space craft
by Michael Foale and Robert Osserman
`https://plus.maths.org/content/right-spin-how-fly-broken-space-craft`

Homework Problems

Problem 10.1. Foucault Pendulum.

A pendulum of mass M that is suspended at a chord of length ℓ in a constant gravity field with acceleration $-g\hat{z}$. We choose Cartesian coordinates for the description of its motion such that the pendulum is at rest in the origin of the coordinate system. The mass of the chord is negligible as compared to its mass such that ℓ is the distance between the pendulum fulcrum and its center of mass.

a) Assume that the pendulum moves in an inertial frame, and that it is performing oscillations with a small amplitude A around its rest position. Sketch the setup.

b) We describe the motion of the pendulum now by only following its x and y coordinate. Determine the kinetic energy T and potential energy V of the pendulum as function of x and y.

c) Perform a Taylor expansion of T and V for small x/ℓ and y/ℓ to show that

$$T \simeq \frac{M}{2}\,\left(\dot{x}^2 + \dot{y}^2\right)^2\,, \qquad\qquad V \simeq \frac{Mg}{2\ell}\,\left(x^2 + y^2\right)^2\,.$$

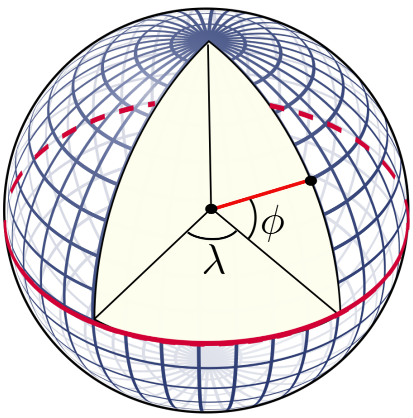
What are the leading order corrections to this equation? Under which condition is it admissible to disregard this correction?

d) Show that the equations

$$\ddot{x} = -\frac{g}{\ell}\,x\,, \qquad\qquad \ddot{y} = -\frac{g}{\ell}\,y\,.$$

provide a faithful description for the motion of the pendulum.
Sketch the solutions in phase space.
Provide the solution for pendulum that is released with zero velocity from a position (x_0, y_0).

e) We explore now how the motion of the pendulum changes when one accounts for the motion of Earth that is spinning with frequency Ω. To this end we take into account now the additional forces acting on the pendulum when it is set up at the latitude ϕ (see sketch to the left). Show that this leads to the following equations of motion



[Peter Mercator, Public Domain, wikimedia]

$$\ddot{x} = -\frac{g}{\ell}\,x + \Omega\,\dot{y}\,\sin\phi$$

$$\ddot{y} = -\frac{g}{\ell}\,y + \Omega\,\dot{x}\,\sin\phi$$

Which forces have been disregarded here? Why is this admissible?

f) Consider now the complex variable $\mathsf{z} = x + \mathrm{i}y$,[1] and demonstrate that its equation of motion is a homogeneous linear differential equation

$$\ddot{\mathsf{z}} + \frac{g}{\ell}\,\mathsf{z} + \Omega\,\dot{\mathsf{z}}\,\sin\phi = 0$$

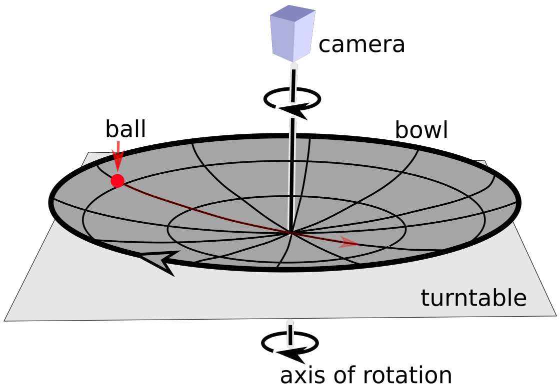
g) Consider the Ansatz $\mathsf{z}(t) = A\,\exp(pt)$ with constant complex numbers p and A to solve the equation of motion. There will be two choices $p_\pm$ that solve the equation. Determine the solution for an initial condition where the pendulum is released at rest from the position $(x_0, 0)$.

[1] Beware the font: The complex variable z must not be confused with the vertical coordinate z!

Demonstrate that the solution describes a pendulum that is swinging in a plane which is slowly rotating around the vertical axis. Determine the rotation frequency.

Problem 10.2. Coriolis forces on the turntable.

The effect of Coriolis forces on a particle can beautifully be described by the motion of a ball moving in bowl placed on a turntable. Let m be the mass of the ball, Ω the rotation frequency of the turntable, and $z(r)$ the cylinder-symmetric height profile of the bottom of the bowl, as function of the distance r from the rotation axis. We will disregard effects due to friction and the rolling of the ball.



a) We select a bowl with a height profile that gives rise to a harmonic force $\vec{F} = -k\vec{r}$ towards the center of the turntable, and we choose Ω such that the centripetal force balances the force $\vec{F}$. Determine Ω.

b) First, we describe the motion of the ball from the perspective of an observer standing at rest next to the turntable. The position of the ball is specified by the complex number $r(t)$, that describes the distance to the rotation axis. Show that the motion of the ball takes the form of an ellipse

$$r(t) = A\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}\,\mathrm{e}^{\mathrm{i}\Omega t} + B\,\mathrm{e}^{\mathrm{i}(\theta-\phi)}\,\mathrm{e}^{-\mathrm{i}\Omega t}\,, \qquad \text{with} \quad \theta, \phi, A, B \in \mathbb{R}\,.$$

How are the parameter of the ellipse related to θ, ϕ, A, and B?

c) Now, we describe the motion of the ball from the perspective of a camera that is mounted on the turntable, and follows the motion in the frame of reference moving with the turntable. In this case we describe the position of the ball by the complex number $q(t)$. Demonstrate that the solution of the equations of motion takes the form

$$\frac{\delta^2 \vec{q}(t)}{\delta t^2} = -2\Omega\,\mathrm{i}\,\frac{\delta \vec{q}(t)}{\delta t}\,.$$

Determine the solution of this equation.

d) At time t_0 we release the ball with velocity $\dot{r}(t_0) = 0$ at the position $r(t_0) = R \in \mathbb{R}$. Determine the resulting solutions $r(t)$ and $q(t)$. How are the solutions related?

Problem 10.3. Throwing balls, the discus, and flipping coins

In the lecture we demonstrated that the evolution of the center of mass of a body and of its spin decouple in force-free flight. At each time during the flight its angular momentum $\vec{L}$ and the rotation axis $\vec{\Omega}$ can then be written as

$$\vec{L} = \sum_\alpha \omega_\alpha(t)\, I_\alpha\, \hat{e}_\alpha(t)\,, \qquad \vec{\Omega}(t) = \sum_\alpha \omega_\alpha(t)\, \hat{e}_\alpha(t) \;\; \text{mit} \;\; \omega_\alpha(t) = \vec{\Omega}(t) \cdot \hat{e}_\alpha(t)\,.$$

Here, the moments of inertia I_α are the eigenvalues of the tensor of inertia

$$I_{ij} = -\int \mathrm{d}^3 q\, \rho(\vec{q})\; q_i\, q_j + \delta_{ij} \int \mathrm{d}^3 q\, \rho(\vec{q})\; (\vec{q} \cdot \vec{q})$$

and $\hat{e}_\alpha$ are the corresponding eigenvectors. These directions rotate with the body and the origin of the coordinate system spanned by the vectors $\vec{q}$ is the center of mass of the body. Their motion is described by the Euler equations

$$\dot{\omega}_1 = -\omega_2\, \omega_3\, \frac{I_3 - I_2}{I_1}\,.$$

and analogous equations for ω_2 and ω_3 that are obtained by cyclic permutation of the indices.

a) Determine the moments of inertia of a solid sphere and of a ping-pong-ball where the mass in concentrated in a thin layer at the surface of the ball (hint: adopt spherical coordinates). Solve the resulting Euler equations. Interpret your result with respect to the rotation of a soccer ball during its flight.

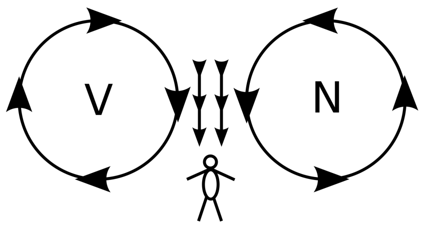
b) Determine the moments of inertia of a cylinder with radius R and height H (hint: adopt cylinder coordinates). We choose $I_1 = I_2 = I$ and I_3 as moment for the rotation around the cylinder axis.

c) A calm throw of the discus is achieved for large ω_3 and $\omega_2 = \omega_1 = 0$. Show that this is a fixed point of the equations of motion. When the throw is not successful the discus wobbles: in that case ω_1 or ω_2 differed from zero when releasing the discus. How does the solution of the Euler equations look in that case? How does this correspond with the wobbling?

d) One can also throw a cylinder differently, as recognized when flipping a coin. Which initial condition does this refer to? How does the solution look like?

Bonus Problem

Problem 10.4. Wind Systems on Earth.



[Mormegil, CC BY-SA 3.0, creativecommons]

On length scales beyond 100 km the mean horizontal wind on Earth is significantly influenced by the Coriolis force. One consequence is expressed by Buys Ballot's law, which was taught to Naval Cadets as:

"In the Northern Hemisphere, if you turn your back to the wind, the low pressure center will be to your left and somewhat toward the front." (Aerology for Pilots, McGraw-Hill, 1943, pg 43)

In the following we explore how the Coriolis force shapes the main wind systems on Earth, and how the Buys Ballot's law comes about.

a) At the equator warm air rises and moves to towards the poles at high altitudes, while cool air moves towards the equator along the ground. We say that the rise of air induces a low pressure region that is sucking air towards the equator. The mean flow is deflected by the Coriolis force. In the vicinity of the equator this gives rise to trade winds.
   From which direction will the winds coming from the North and from the South approach the equator?

b) The velocity, i.e. the speed and the direction, of the flow will no longer change when the acceleration of the wind by the pressure gradient and by the Coriolis force balance (geostrophic wind). We consider the Euler equation for the momentum-balance of fluid flow to explore this relation

$$\varrho \, \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P$$

   In this equation ϱ is the mass density of air, $\vec{u}$ is the flow velocity, and P is the pressure. The equation holds in inertial systems and when the viscosity of the fluid may be neglected. The latter condition holds for the atmosphere. The former condition entails that the time derivative must be augmented by the Coriolis force when adopting a coordinate system that is co-moving with the Earth surface. Demonstrate that condition for a stationary flow velocity $\vec{u}$ amounts then to

$$\nabla P = -2\varrho\,\vec{\Omega} \times \vec{u}$$

Here, $\vec{\Omega}$ is the angular velocity of Earth.

c) Determine the pressure difference over a distance of 1000 km when air is moving at a (mean) horizontal speed of 50 km/h. How does the pressure difference depend on latitude? What is the relation between the direction of the pressure gradient and the flow velocity?
Hint: Air has a density of about $\varrho \simeq 1.3\,\mathrm{kg/m^3}$.

d) Consider now the flow along the equator. How does the pressure change upon motion to the West and to the East, respectively? What does this tell about the stability of the wind? Why is this argument incomplete at best and wrong if worst comes to worst?

e) High pressure and low pressure regions at mid latitudes (for instance close to Europe) have typical diameters of 1000 km, and the predominant wind directions are along isobars rather than in the direction of the pressure gradient. Compare the pressure difference determined in c) with typical pressure difference of high pressure and low pressure regions, and discuss the orientation of the flow around the respective regions.

I wish everybody a merry holiday season,
looking forward to see you back in Januray.