Homework Exercises 9

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Dec 16, at 11:00,
or in my mail box **in the Linnéstr**, by Monday, Dec 16, 13:00.
Please submit only Problem 9.3 when you participate in the seminar.

Warm-up

Problem 9.1. Planetary Conjunctions.

Every now and then there are planetary conjunctions, where two or even several planets appear in a very close vicinity on the sky. The conjunction of Mercury and Venus appearing above the Moon is shown to the right. Right at the moment you can observe a conjunction of Venus and Saturn, and upcoming events are listed on wikipedia. A funny feature of conjunctions is that the times between subsequent conjunctions vary a lot. For instance there have been recent Mercury-Venus conjunctions recently on September 13 and October 30, 2019, while the forthcoming conjunctions will only arise on May 22, 2020.

6 March 2008 [ESO/Y. Beletsky [CC BY 4.0]

a) Determine the time evolution of the angle of sight of the planet, as they are observed at midnight.

b) Discuss the monotinicity of the evolution of the planet position on the night sky.

c) What does this dependences imply about the time intervals between subsequent conjunctions of a pair of stars? Can you derive a recusion relation for the times?

d) Plot subsequent times as function of one another. What do you observe?

Homework Problems

Problem 9.2. Falling through Earth.

Assume that a powerful wizard magics away the electromagnetic interaction of my body with the Earth. Where and when will I reappear on the Earth surface?

To deal with this problem we adopt the approximation that the Earth is spherical with radius R, and that it has a constant mass density ρ. Let my mass be m. I will then only interact with the Earth by the graviational interaction, that gives rise to a force

$$\vec{F}(\vec{q}) = \frac{\rho\, m\, G\, \vec{q}}{|\vec{q}|^3}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

for volume elements $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ at a position $\vec{q}$ relative to my position.

a) In contrast to the convention adopted in our treatment of the Kepler problem the equation for the force given above has a positive sign. Why is this meaningful here?

b) Let $\vec{r}$ be my position as seen from the center of Earth. Verify that the net force that is acting on me can then be written as

$$\vec{F}(\vec{r}) = \rho\, m\, G\, \hat{r} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \mathrm{d}\theta \int_{-q_-(\theta)}^{q_+(\theta)} \mathrm{d}q\ \sin\theta\, \mathrm{sign}(q)$$

where

$$q_\pm(\theta) = \pm|\vec{r}|\ \cos\theta + \sqrt{R^2 - |\vec{r}|^2\ \cos^2\theta}$$

and θ is the angle between $\vec{r}$ and $\vec{q}$.

Hint: These are spherical coordinates where $\vec{r}$ is aligned with the $\theta = 0$ axis.

c) Perform the integration. Observe subsequently that at the Earth surface there is a gravitational acceleration $\vec{g}$ acting on my body. Use this special case to eliminate G and ρ from your result. You should thus obtain

$$\vec{F}(\vec{r}) = -mg\frac{\vec{r}}{R}$$

d) At the moment of my departure I am at rest with respect to the Earth surface. What does that imply for the initial conditions of my equation of motion?

e) Demonstrate that my motion will be constrained to a plane. Based on my initial condition I choose a coordinate system in this plane where z is vertically up in the direction of gravity, and w denotes West. Thus, my initial position in this coordinate system is $(z_0, w_0) = (R, 0)$. What will be my position at t after departure? What will be my closest approach to the center of Earth? Where and when will I reappear on the Earth surface?

Problem 9.3. Coulomb potential and external electric forces.
We consider the Hydrogen atom to be a classical system as suggested by the Sommerfeld model. Let the proton be at the center of the coordinate system and the electron at the position $\vec{r}$. The interaction between the proton and the electron is described by the Coulomb potential $\alpha/|\vec{r}|$. In addition to this interaction there is a constant electric force acting, that is described by the potential $\vec{V} \cdot \vec{r}$. Altogether the motion of the electron is therefore described by the potential

$$U = -\frac{\alpha}{|\vec{r}|} - \vec{F} \cdot \vec{r}$$

a) Sketch the system and the relevant parameters.

b) Which force is acting on the particle? How do its equation of motion look like?

c) Verify that the energy is conserved.

d) Show that also the following quantity is a constant of motion,

$$I = \vec{F} \cdot \left(\dot{\vec{r}} \times \vec{L} \right) - \alpha \, \frac{\vec{F} \cdot \vec{r}}{|\vec{r}|} + \frac{1}{2} \, \left(\vec{F} \times \vec{r} \right)^2$$

Here $\vec{L}$ is the angular momentrum of the particle with respect to the origin of the coordinate system.

Problem 9.4. Surface area of a hypersphere.

A hypersphere is a generalization of a sphere to d-dimensional space. The surface of a d-dimensional hypersphere with radius R comprises all points $\vec{q} \in \mathbb{R}^d$ with $|\vec{q}| = R$.
For $d = 2$ this is a circles and its surface "area" amounts to $A_2 = 2\pi R$.
For $d = 3$ this is a normal sphere with area $A_3 = 4\pi R^2$.
In general the surface area can be written as $A_d = S_d\, R^d$ where S_d is the surface area of the d-dimensional unit sphere.
We will adopt hyper-spherical coordinates to evaluate the d-dimensional Gaussian integral

$$G_d = \int_{\mathbb{R}^d} \mathrm{d}^d q \;\exp(-\vec{q}^2) = \int_{-\infty}^{\infty} \mathrm{d}q_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}q_d \;\exp(-\sum_{i=1}^{d} q_i^2)$$

a) Proof that $G_d = (G_1)^d$.

b) Consider the case $d = 2$, and introduce polar coordinates to show that

$$\left[\int_{-\infty}^{\infty} \mathrm{d}q \;\exp(-q^2)\right]^2 = 2\pi \int_0^{\infty} \mathrm{d}q\; q \;\exp(-q^2)$$

Solve the integral on the right-hand-side by adopting the substitution $w = q^2$. What does this tell about G_1?

c) Rewrite the intgral for $d = 3$ in the form

$$\pi^{3/2} = G_1^3 = G_3 = S_3 \int_0^{\infty} \mathrm{d}q\, q^2 \;\exp(-q^2)$$

and use partial integration to show that this entails $S_3 = 4\pi$.

d) Generalize the previous argument to the d-dimensional case, and adopt induction to proof that

$$S_{d-1} = \frac{d\, \pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}$$

with $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(x + 1) = x\, \Gamma(x)$.