Homework Exercises 8

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Dec 10, at 11:00,
or in my mail box at ITP, room 105b, by Monday, Dec 10, 13:00.

Warm-up

Problem 8.1. Car on a air-cushion

We consider a car of mass $m = 20\,\mathrm{g}$ moving – to a very good approximation without friction – on an air-cushion track. There is a string attached to the car that moves over a roll and hangs vertically down on the side opposite to the car.

a) Sketch the setup and the relevant parameters.

b) Which acceleration is acting on the car when the string is vertically pulled down with a force of $F = 2\,\mathrm{N}$. Determine the volocity $v(t)$ and its position $x(t)$.

c) Determine the force acting on a $200\,\mathrm{g}$ chocolate bar, in order to get a feeling for the size of the force that was considered in (b).

d) Now we fix the chocolate bar at the other side of the string. The equation can then be obtained based on energy conservation

$$E = E_{\mathrm{kin}} + E_{\mathrm{pot}} = \frac{m+M}{2}\,v^2 + Mgh = \mathrm{konst},$$

where M is the mass of the chocolate bar. Is the acceleration the same of different as in the cases (b) and (c)? Provide an argument for your conclusion.

Homework Problems

Problem 8.2. Motion in a harmonic central force field

A particle of mass m and at position $\vec{r}(t)$ is moving under the influence of a central force field

$$\vec{F}(\vec{r}) = -k\,\vec{r}\,.$$

a) We want to use the force to build a particle trap,[1] i.e. to make sure that the particle trajectories $\vec{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\vec{r}(t)| < B$ for all times t. What is the requirement on k to achieve this aim?

b) Determine the energy of the particle and show that the energy is conserved.

c) Demonstrate that the angular momentum $\vec{L} = \vec{r} \times m\,\dot{\vec{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

d) Let (x_1, x_2) be the coordinates in the plane that is singled out by the angular momentum conservation. Show that $m\ddot{x}_i(t) + k\,x_i(t) = 0$ for $i \in \{1, 2\}$. Determine the solution of these equations. Sketch the trajectories in the phase space $(x_i, \dot{x}_i)$. What determines the shape of the trajectories?

e) Show that the trajectories in the configuration space (x_1, x_2) are ellipses. What determines the shape of these trajectories?

f) Discuss the relation between the amplitude and shape of the trajectory, as determined by the ratio and the geometric mean of the major axes of the ellipse in configuration space, and the period of the trajectory.

[1]Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

Problem 8.3. Motion of two particles with harmonic interaction

We consider the motion of two particles of masses m_1 and m_2 at positions $q_1(t)$ and $q_2(t)$. They interact with a harmonic force, such that

$$\begin{aligned} m_1\, \ddot{\vec{q}}_1(t) &= -k\left(\vec{q}_1(t) - \vec{q}_2(t)\right), \\ m_2\, \ddot{\vec{q}}_2(t) &= -k\left(\vec{q}_2(t) - \vec{q}_1(t)\right). \end{aligned}$$

a) Discuss the evolution of the center of mass of this two particle system. How does it evolve in time, and how does the evolution depend on the initial condition $\big(\vec{q}_1(t_0), \vec{q}_2(t_0), \dot{\vec{q}}_1(t_0), \dot{\vec{q}}_2(t_0)\big)$?

b) Determine the equations of motion of the relative motion $\vec{R}(t) = \vec{q}_1(t) - \vec{q}_2(t)$. Discuss their solution by adapting the results of Problem 8.2.

c) How to trajectories of the two particles with harmonic interaction differ from those of the Kepler problem?