Homework Exercises 7

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Dec 2, at 11:00,
or in my mail box at ITP, room 105b, by Monday, Dec 2, 13:00.
Please submit only Problem 7.5 when you participate in the seminar.

Warm-up

Problem 7.1. Pulling a Duck.

A child is pulling a toy duck with a force of $F = 5\,\mathrm{N}$. The duck has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.[1]

a) Describe the motion of the duck when there is no friction.
In the beginning the duck is at rest.

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the duck.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

Problem 7.2. Crossing a River.

A ferry is towed at the bank of a river of width $B = 100$ m that is flowing at a velocity $v_F = 4$ m/s to the right. At time $t = 0$ s it departs and is heading with a constant velocity $v_B = 10$ km/h to the opposite bank.

a) When will it arrive at the other bank when it always heads straight to the other side? (In other words, at any time its velocity is perpendicular to the river bank.)

How far will it drift downstream on its journey?

[1]For this angle one has $\tan\theta \approx 3/4$.

b) In which direction (i.e. angle of velocity relative to the downstream velocity of the river) must the ferryman head to reach exactly at the opposite side of the river?

Determine first the general solution. What happens when you try to evaluate it for the given velocities?

Problem 7.3. Running Mothers.

In the Clara Zetkin Park one regularly encounters blessings[2] of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their youngs in pouches.

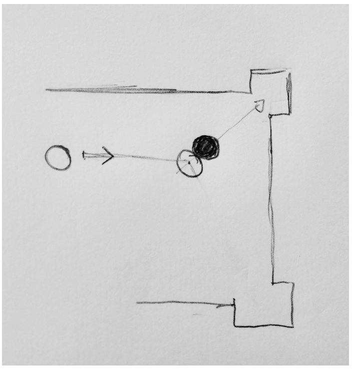
a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.
The carriages suffer from friction. Let the friction coefficient be $\gamma = 0.3$.
When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.

b) How does the running speed matter in this discussion?

c) How does the mass of the babies/youngs make a difference?

Problem 7.4. Collisions on a billiard table.

The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity $\vec{v}$), and hit the black ball such that it ends up in pocket to the top right. What is tricky about the sketched track? What might be a better alternative?

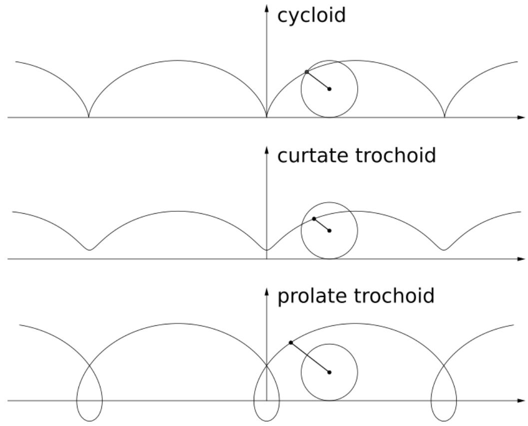


[2]Look up "terms of venery" if you ever run out of collective nouns.

Homework Problems

Problem 7.5. Retroreflector paths on bike wheels.

The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retroreflectors fixed in the sparks enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio ρ of the reflectors distance d to the wheel axis and the wheel radius r.



[Wikimedia CC BY 4.0 (modified)]

A small stone in the profile traces a cycloid ($\rho = 1$). Animations of the trajectories can be found at `https://en.wikipedia.org/wiki/Trochoid` and `http://katgym.by.lo-net2.de/c.wolfseher/web/zykloiden/zykloiden.html`.
A trochoid is most easily described in two steps: Let $\vec{M}(\theta)$ be the position of the center of the disk, and $\vec{D}(\theta)$ the vector from the center to the position $\vec{q}(\theta)$ that we follow (i.e. the position of the retroreflector) such that $\vec{q}(\theta) = \vec{M}(\theta) + \vec{D}(\theta)$.

a) The point of contact of the wheel with the street at the initial time t_0 is the origin of the coordinate system. Moreover, we single out one spark and denote the change of its angle with respect to its initial position as θ. Note that negative angles θ describe forward motion of the wheel!

   Sketch the setup and show that

$$\vec{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix}, \qquad \vec{D}(\theta) = \begin{pmatrix} -d\,\sin(\varphi + \theta) \\ d\,\cos(\varphi + \theta) \end{pmatrix}.$$

   What is the meaning of φ in this equation?

b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time, $L = \int_{t_0}^{t} \mathrm{d}t \, \left|\dot{\vec{q}}(\theta(t))\right|$. Show that therefore

$$L = r \int_0^\theta \mathrm{d}\theta \, \sqrt{1 + \rho^2 + 2\rho\,\cos(\varphi + \theta)}$$

c) Consider now the case of a cycloid and use $\cos(2x) = \cos^2 x - \sin^2 x$ to show that the expression for L can then be written as

$$L = 2\,r \int_0^\theta \mathrm{d}\theta \, \left| \cos \frac{\varphi + \theta}{2} \right|$$

How long is one period of the track traced out by a stone picked up by the wheel profile?

Problem 7.6. Elastic two-particle collisions.

We consider the positions of two balls, $i \in \{1, 2\}$, with masses m_i and radii R_i at positions $\vec{q}_i \in \mathbb{R}^3$. In the beginning of the experiments, at time t_0, they have velocities $\dot{\vec{q}}_i(t_0) = \vec{v}_{0,i}$.

a) Determine the evolution of the center of mass $\vec{Q}(t)$ of the two balls. How does it evolve when the two balls are thrown on a playground, with gravitational acceleration $\vec{g} = -g\hat{z}$ acting?

b) Consider now the motion of the two balls relative to their center of mass (CM). Let these positions be $\vec{r}_i(t) = \vec{q}_i(t) - \vec{Q}(t)$ and the associated momentum be $\vec{p}_i(t) = m_i\,\dot{\vec{r}}_i(t)$. Show that the following relations hold for the motion relative to the center of mass, irrespective of the choice of initial conditions

$$\vec{p}_{\mathrm{rel}} = \vec{p}_1 + \vec{p}_2 = \vec{0} \qquad \text{and} \qquad \vec{L}_{\mathrm{rel}} = (\vec{r}_2 - \vec{r}_1) \times \vec{p}_2 \,.$$

c) Compare the result of (b) with the relations that we discussed in the lecture for the case of two disks that were not subjected to an external gravitational force. How do the results obtained in the lecture carry over to the present system?
Hint: Most effectively this is answered by showing that for all times the vectors $\vec{r}_2$ and $\vec{v}_2$ are orthogonal to $\vec{L}_1 = \vec{r}_1 \times \vec{p}_1$, and that therefore $\vec{r}_1$, $\vec{v}_1$, $\vec{r}_2$, and $\vec{v}_2$ will always lie in a plain.

d) How do the trajectories look like before the collision (in the CM system and for an observer standing on the playground).

e) How do the momenta of the particles change in an elastic collision? How do the trajectories look like after the collision (in the CM system and for an observer standing on the playground).

Problem 7.7. Inelastic collisions, ballistics, and cinema heroes.

We first discuss a few CSI techniques to investigate firearms. Then we wonder how cinema heroes shoot.

a) The velocity of a projectile can be determined by investigating its impact into a wooden block (mass M) that is fixed to a swing with arms of length ℓ. We choose our coordinates such that gravity acts in negative z-direction, and the block moves in the $(x - z)$-plane. The swing fulcrum is at the origin of the coordinate system. The angle θ describes the angle of the arm with respect to the negative z axis.

   We set up the experiment such that initially $\theta = 0$ while the projectile approaches the wooden block along a trajectory parallel to the x-axis. At time t_0 it hits the center of the block. The projectile has mass m and velocity $\vec{v}_0$. Sketch the setup.

b) What is the angular momentum of the projectile, the wooden block and the total angular momentum before the impact of the bullet? How does it change upon impact? What is the angular momentum of the swing after the impact? What is its kinetic and potential energy? How far does it rise until it reverses its direction of motion? For which projectile velocities does it to over the fulcrum? Bonus: Determine the kinetic energy of the projectile before the impact and compare it to the kinetic energy of the swing immediately after the impact. What is the origin of the energy difference?

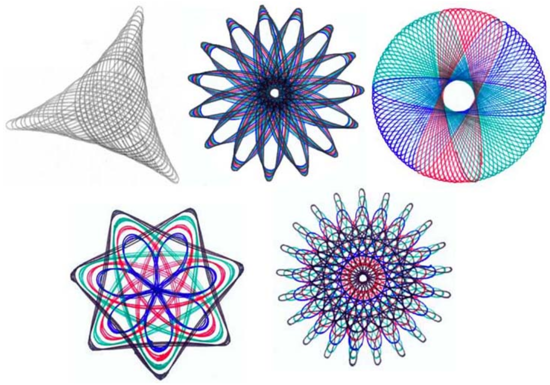
c) The title of Stanley Kubrick's movie *Full Metal Jacket* refers to full metal jacket bullets, i.e. projectiles of the M16 assault rifle used in the Vietnam war. Its bullets have a mass of 10 g and they deflect a 1 kg wooden block suspended at a 2 m swing arm to a maximum angle of 127°. What is the velocity of the bullets?

   The bullets of a 9 mm Luger pistol have a mass of 8 g and they are fired with a muzzle velocity of 350 m/s. What is the associated the maximum deflections of the swing?

d) What does this tell about the recoil of the pistol and the rifle?
   Have a look now at the Rambo shooting scene on YouTube where you see him performing a 30seconds burst fire with about 200rounds/minute.
   Estimate the force needed to return the rifle to the initial position by the next shot, and the resulting amplitude of the quivering of the gunman's arm. What do you conclude about this scene?

Bonus Problem

Problem 7.8. Hypotrochoids, roulettes, and the spirograph.



[Wikimedia Public domain]

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

a) Consider the track of a pole attached to a disk with n cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the curves developed in Problem 7.5 such that you can describe hypotrochoids.

c) Test your result by writing a Python program that plots the cruves for given m and n.