Homework Exercises 6

Your solution to the problems should be handed in/presented
either during a seminar on Monday, Nov 25, at 11:00,
or in my mail box at ITP, room 105b, by Monday, Nov 25, 13:00.
Please submit only Problem 6.5 when you participate in the seminar.

Warm-up

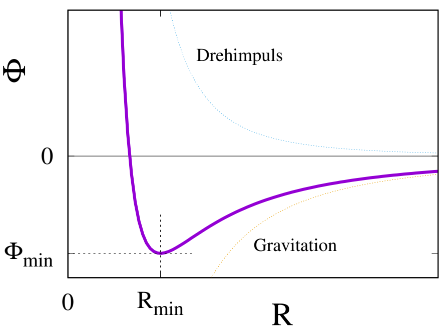
Problem 6.1. Phase-space portraits for a scattering problem.

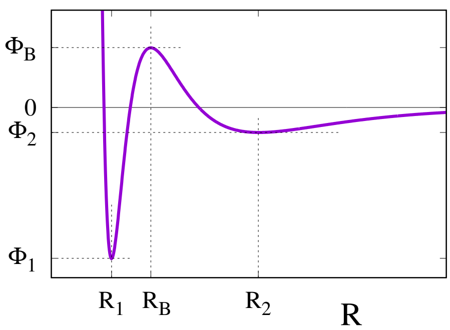
Sketch the potential $\Phi(x) = 1 - 1/\cosh x$ for $x \in \mathbb{R}$. Add to the sketch a the phase portait of the motion in this potential, i.e. the solutions of $\ddot{x} = -\partial_x \Phi(x)$, in the phase space $(x, \dot{x})$.

Homework Problems

Problem 6.2. Phase-space portraits for the Kepler and the DLVO problem.

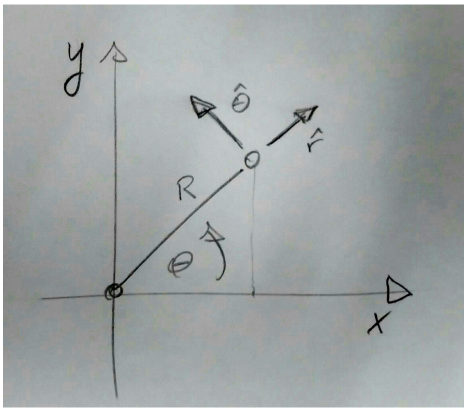
The figures below show the effective potentials for the distance between two planets in the Kepler problem, and for the DLVO potential for the interaction of charged colloids.[1] Sketch the solutions for classical trajectories in these potentials in the phase space $(R, \dot{R})$.





[1] The DLVO theory predicts that there are two distinct avarage bond length for aggregates of two colloids. There is a strong bond of strength Φ_1 where the colloids have a small bond length R_1, and a weach bond of strength Φ_2 at a larger distance R_2. Between these two states there is an energy barrier of height Φ_B.

Problem 6.3. Motion on a circular track.



The position of a particle in the plane can be specified by Cartesian coordinates (x, y) of by polar coordinates with basis vectors $\hat r(\theta)$ and $\hat\theta(\theta)$, that have the following representation in Cartesian coordinates (cf. the sketch to the left)

$$\hat r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

We will now explore the trajectory $\vec q(t)$ of a particle with mass m that moves on a track with a fixed radius R.

a) Verify that $\hat\theta = \partial_\theta \hat r$ and $\partial_\theta^2 \hat r = -\hat r$.
Please provide a geometric interpretation of this result!

b) The position of the particle can be specified as $\vec q(t) = R\,\hat r(\theta(t))$. Determine $\dot{\vec q}$ and $\ddot{\vec q}$ based on this equations. Verify your result by performing the same calculation in Cartesian coordinates.

c) Which force is required to keep the particle on the circular track? What does this imply for curves in bike races, bobsled races and in skate parks?

d) Consider the motion at a constant angular velocity, $\theta(t) = \omega t$, and show that the acceleration in this setting takes the form $\ddot{\vec q} = -R\,\omega^2\,\hat r(\omega t)$ Verify that this amounts to a force that is perpendicular to the velocity. What does this imply for the absolute value of the velocity?
Hint: Discuss the time derivative of $\vec v^2$.

Problem 6.4. Free fall with viscous friction.

The falling of a ball in a viscous medium can be described by the equations of motion

$$\ddot h(t) = -g - \gamma\,\dot h(t)\,.$$

Here $h(t)$ is the vertical position of the ball, g is the acceleration due to gravity, and the coefficient $\gamma \simeq \frac{2}{3}\eta/\rho_{\text{ball}} R^2$ describes the viscous drag. Here R is the radius of the sphere, ρ_{ball} is the mass density of its material, and η is the viscosity of the surrounding fluid. For air and water it takes values of about $\eta_{\text{air}} \simeq 2 \times 10^{-5}\,\text{kg/m s}$, and $\eta_{\text{water}} \simeq 1 \times 10^{-3}\,\text{kg/m s}$, respectively.

a) Argue that $w(\tau) = \gamma \dot{h}(t)/g$ with $\tau = \gamma t$ obeys the equation

$$\frac{\mathrm{d}w(\tau)}{\mathrm{d}\tau} = -1 - w(\tau)\,.$$

How do you recover the the dependence of the motion on the parameters g and γ?

b) Determine the solution of the equation for the initial condition $w(\tau_0) = w_0$. What happens for $w_0 = -1$?

c) Determine $h(t)$ for a ball that is released at a height H with zero velocity, and with an upward velocity of v_0.

d) Sketch the solution $h(t)$. How does the solution look like for small and for large t? In particular: Determine the Taylor expansion for the trajectory to third order in t. How does it differ from a free fall with $\gamma = 0$?

e) Estimate the time scale where the viscosity does not yet lead to noticeable differences from a description with viscosity of a bullet with a diameter of 1 cm, when it drops down from the balcony and when it is vertically shot into the air with initial velocity 100 m/s. How far did it travel in that time?

f) How do the conclusions change for a harpoon shot under water? (Assume for simplicity that it is sufficient to treat it like a ball with radius corresponding to the diameter of the arrow.)

Problem 6.5. Free fall with turbulent friction.

For large velocities the motion of the fluid around the ball goes turbulent, and the friction crosses over to a drag force with modulus

$$F_D = \frac{\rho_{\text{fluid}}\, \pi R^2\, C_D}{2}\, v^2 = \kappa\, v^2$$

where the drag coefficient typically takes values between 0.5 and 1. A very beautiful description of the physics of this equations has been provided in an instruction video by the NASA (click here to check it out).

a) Argue that $u(\tau) = \sqrt{\kappa/g}\, \dot{h}(t)$ with $\tau = \sqrt{\kappa g}\, t$ obeys the equation

$$\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} = -1 - u^2(\tau)\, \operatorname{sign}\big(u(\tau)\big)\,.$$

How do you recover the dependence of the motion on the parameters g and κ?

b) Determine the solution of the equation for the initial condition $u(\tau_0) = u_0$. What happens for $u_0 = -1$?[2]

c) Determine $h(t)$ for a ball that is released at a height H with zero velocity, and with an upward velocity of v_0.

d) Do you trust the solution for zero velocity? If not: provide an estimate for the range of parameters and initial conditions where the solution applies.

Bonus Problem

Problem 6.6. Maximum distance of flight.

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

Challenge. What is the optimum choice of θ for the shot put?

c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

[2] **Hint:** The sign-function is best handled as follows. Start by dealing with initial conditions where $u_0 \leq 0$, and show that the velocity will then always remain negative. In this case $\text{sign}(u(\tau)) = -1$ for all times. Subsequently, explore initial conditions with $u_0 > 0$. For these initial conditions you look for solutions where $u(\tau)) > 0$. At some time τ_* and position z_* the velocity becomes negative, however. The trajectory will then continue according to a trajectory started at time τ_* at position z_* with velocity zero. This evolution we know, however.