Homework Exercises 4

Groups

Problem 4.1. Checking Group Axioms
Which of the following sets are groups?

a) $(\mathbb{N}, +)$

b) $(\mathbb{Z}, +)$

c) $(\mathbb{Z}, \cdot)$

d) $(\{+1, -1\}, \cdot)$

e) $(\{0\}, +)$

$(\{1, \ldots, 12\}, \oplus)$

where $\oplus$ in f) refers to adding as we do it on a clock, e.g. $10 \oplus 4 = 2$.

Trigonometry

Problem 4.2. Euler's Equation and Trigonometric Relations
Euler's equation $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$ relates complex values exponential functions and trigonometric functions.

a) Sketch the position of $\mathrm{e}^{\mathrm{i}x}$ in the complex plain, and indicate how Euler's equation is related to the Theorem of Pythagoras.

b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular one has $\mathrm{e}^{\mathrm{i}\,(x+y)} = \mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{\mathrm{i}y}$. Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about $\sin(2x)$ and $\cos(2x)$?

Problem 4.3. Properties of Right-Angled Triangles

a) Fill in the gaps for the values of the angle θ in radians, and employ the symmetry of the trigonometric sine and cosine functions to determine the values in the right columns

θ		$\sin\theta$	$\cos\theta$
$\circ$	rad		
0	--------	0	--------
30	--------	$\frac{1}{2}$	--------
45	--------	$\frac{\sqrt{2}}{2}$	--------
60	--------	$\frac{\sqrt{3}}{2}$	--------
90	--------	1	--------
120	--------	--------	--------
135	--------	--------	--------
150	--------	--------	--------
180	--------	--------	--------

b) Consider a right triangle where one of the angles is θ. How are the length of its sides related to $\sin\theta$ and $\cos\theta$? Check that the Theorem of Pythagoras holds! Do you see a systematics for the values provided for $\sin\theta$?

c) Use the symmetries of the trigonometric functions to determine the values provided for $\theta = \pi/4$.

Use the symmetries of the trigonometric functions and the trigonometric relation for $\sin(2\theta)$ to determine the values provided for $\theta = \pi/6$ and $\theta = \pi/3$.

The values for $\pi/10$, $\pi/8$, and $\pi/5$ can also be stated explicitly in elementary form. Determine the expressions for these values!

Differentiation

Problem 4.4. Derivatives of Elementary Functions
Determine the derivatives of the following functions.

a) $\sin x$

b) $\cos x$

c) $\tan x$

d) x^n

e) $\sinh x$

f) $\cosh x$

g) $\tanh x$

h) e^x

i) $\ln x$

$\log_{10} x$

n^x

x^x

Problem 4.5. Derivatives of Common Composite Expressions
Evaluate the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}(a+x)^b$

b) $\frac{\partial}{\partial x}(x+b\,y)^2$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x+y(x))^2$

d) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta(t)$

e) $\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin\theta(t)\ \cos\theta(t)\big)$

f) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\big(2\theta(t)\big)$

g) $\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{a+b\,z^2}$

h) $\frac{\partial}{\partial x_3}\left[\sum_{j=1}^{6} x_j^2\right]^{-1/2}$

i) $\frac{\partial}{\partial y_1}\ln(\vec{x}\cdot\vec{y})$

In these expressions a and b are real constants, and $\vec{x}$ and $\vec{y}$ are 6-dimensional vectors.

Integration

Problem 4.6. Integrals of Elementary Functions
Evaluate the following integrals.

a) $\displaystyle\int_{-1}^{1} \mathrm{d}x\,(a+x)^2$

b) $\displaystyle\int_{-5}^{5} \mathrm{d}q\,(a+b\,q^3)$

$\displaystyle\int_{0}^{B} \mathrm{d}k\ \tanh^2(kx)$

c) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\,\mathrm{e}^{-x/L}$

d) $\displaystyle\int_{-L}^{L} \mathrm{d}y\,\mathrm{e}^{-y/\xi}$

e) $\displaystyle\int_{0}^{L} \mathrm{d}z\,\frac{z}{a+b\,z^2}$

f) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\,x\,\mathrm{e}^{-x^2/(2Dt)}$

g) $\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}\ell\,\ell\,\mathrm{e}^{-\ell^2/(2Dt)}$

$\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}z\,x\,\mathrm{e}^{-z\,x^2}$

Except for the integration variable all quantities are considered to be constant.
Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

Problem 4.7. Solving Integrals by Partial Integration
Evaluate the following integrals by partial integration

$$\int \mathrm{d}x\, f(x)\, g'(x) = f(x)\, g(x) - \int \mathrm{d}x\, f'(x)\, g(x)$$

a) $\displaystyle\int_a^b \mathrm{d}x\, x\, \mathrm{e}^{kx}$ b) $\displaystyle\int_a^b \mathrm{d}x\, x^2\, \mathrm{e}^{kx}$ $\displaystyle\int_a^b \mathrm{d}x\, x^n\, \mathrm{e}^{kx},\ n \in \mathbb{N}$

The integral c) can only be given as a sum over $j = 0, \ldots, n$.

Problem 4.8. Substitution with Trigonometric and Hyperbolic Functions
Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{q(x_1)}^{q(x_2)} \mathrm{d}q\, f(q) = \int_{x_1}^{x_2} \mathrm{d}x\, q'(x)\, f(q(x))$$

with a function $q(x)$ that is bijective on the integration interval $[x_1, x_2]$.

a) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}}$ by substituting $x = \sin\theta$

b) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}}$ by substituting $x = \sinh z$

c) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1+x^2}$ by substituting $x = \tan\theta$

d) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1-x^2}$ by substituting $x = \tanh z$

Vectors and Coordinates

Problem 4.9. Carthesian Coordinates

a) Mark the following points in a carthesian coordinate system:

$$(0;0) \quad (0;3) \quad (2;5) \quad (4;3) \quad (4;0)$$

Add the points $(0;0)$ $(4;3)$ $(0;3)$ $(4;0)$, and connect the points in the given order. What do you see?

b) What do you find when drawing a line segment connecting the following points?

$$(0;0) \quad (1;4) \quad (2;0) \quad (-1;3) \quad (3;3) \quad (0;0)$$

Problem 4.10. Eagle and Hedgehog

a) You are looking from South into a valley and see a hedgehog that sits 10 m to the right of a tree on in the grass. From far right above an eagle is attacking. You monitor its position based on coordinates to the right and vertical from the foot of the tree, indicating distances in meter:

$$(10.9\,;\,28) \quad (11.5\,;\,44) \quad (12.1\,;\,62) \quad (13.2\,;\,99)$$

Will the eagle catch the hedgehog?

b) A friend is taking a movie of the same incidence, observing it from the East. From the movie you extract the position of the eagle to the North and vertical with respect ot the foot of the tree.

$$(1.9\,;\,27) \quad (2.4\,;\,44) \quad (3.1\,;\,62) \quad (3.8\,;\,85)$$

Will the eagle catch the hedgehog?

c) A forester, who knows the animals really well, is telling you that the hedgehog will typically start moving North with a speed of about $1\,\mathrm{m\,s^{-1}}$ when he notices the eagle coming down. Will the eagle catch the hedgehog when it is coming down with a speed of $50\,\mathrm{m\,s^{-1}}$ and when the hedgehog notices the eagle at a heigt of 50 m?