Homework Exercises 3

Your solution to the problems 3.3–3.5 should be handed in/presented
either during a seminar on Monday, Nov 4, at 11:00,
or in my mail box at ITP, room 105b, by Monday, Nov 4, 13:00.
Please submit only Problem 3.5 (a)–(e) when you participate in the seminar.

Warm-up

Problem 3.1. Angles between three balanced forces
Consider three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_p$ like in the rubber band example of the lecture, where I pull the band with force $\vec{F}_p$ and this force is balanced by the forces due to the tension of the rubber band.

a) Make a sketch of the setup where you indicate the angels $\angle(\vec{F}_p, \vec{F}_1)$ as θ_{1p} and $\angle(\vec{F}_p, \vec{F}_2)$ as θ_{2p}, respectively.

b) Determine the condition for a balance of forces in the directions parallel to $\vec{F}_p$ and parallel to $\vec{F}_1$.

c) The result of (b) can be expressed as a conditions on $F_p = |\vec{F}_p|$ as function of F_1, F_2, θ_{1p} and θ_{2p}, and on F_1 as function of F_p, F_2, θ_{1p} and θ_{2p}. Insert the former condition into the latter one in order to eliminate F_p.
Hence, you find that F_1 will be proportional to F_2, when the angles θ_{1p} and θ_{2p} are fixed. What does this reflect from a physical point of view?
Hint: What happens to the force balance when you fix the angle and increase $\vec{F}_p$ by a factor φ.

d) Employ trigonometric relations to show that the proportionality constant can be written as a ratio of two sines, i.e. one has

$$F_1 = \frac{\sin\alpha}{\sin\alpha}\, F_2$$

How are the angles α and β related to θ_{1p} and θ_{2p}?

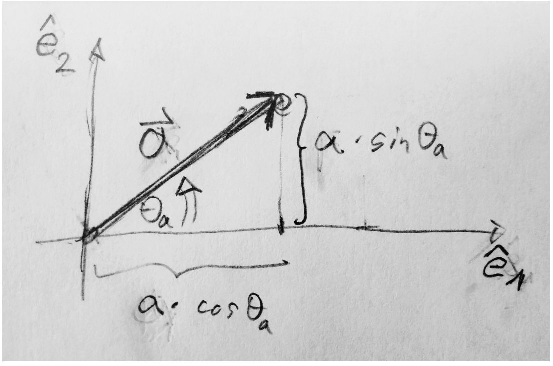
e) Can you find a simpler way to derive the expression found in (d)?

Problem 3.2. Geometric and algebraic form of the scalar product

The sketch to the right shows a vector $\vec{a}$ in the plane, and its representation as a linear combination of two orthonormal vectors $(\hat{e}_1, \hat{e}_2)$,

$$\vec{a} = a\,\cos\theta_a\,\hat{e}_1 + a\,\sin\theta_a\,\hat{e}_2$$

Here, a is the length of the vector $\vec{a}$, and $\theta_1 = \angle(\hat{e}_1, \vec{a})$.



a) Analogously to $\vec{a}$ we will consider another vector $\vec{b}$ with a representation

$$\vec{b} = b\,\cos\theta_b\,\hat{e}_1 + b\,\sin\theta_b\,\hat{e}_2$$

Employ the rules of scalar products, vector addition and multiplication with scalars to show that

$$\vec{a}\cdot\vec{b} = a\,b\,\cos(\theta_a - \theta_b)$$

Hint: Work backwards, expressing $\cos(\theta_a - \theta_b)$ in terms of $\cos\theta_a$, $\cos\theta_b$, $\sin\theta_a$, and $\sin\theta_b$.

b) As a shortcut to the explicit calculation of a) one can introduce the coordinates $a_1 = a\,\cos\theta_a$ and $a_2 = a\,\sin\theta_a$, and write $\vec{a}$ as a tuple of two numbers. Proceeding analogously for $\vec{b}$ one obtains

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

How will the product $\vec{a}\cdot\vec{b}$ look like in terms of these coordinates?

c) How do the arguments in a) and b) change for D dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors $\hat{e}_1, \ldots, \hat{e}_D$?

What changes when the basis is not orthonormal?
What if it is not even orthogonal?

Homework Problems

Problem 3.3. Linear Dependence of three vectors in 2D

In the lecture I pointed out that every vector $\vec{v} = (v_1, v_2)$ of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors $\vec{a}$ and $\vec{b}$,

$$\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b}$$

In this exercise we revisit this statement for $\mathbb{R}^2$ with the standard forms of vector addition and multiplication by scalars.

a) Provide a triple of vectors $\vec{a}$, $\vec{b}$ and $\vec{v}$ such that $\vec{v}$ can *not* be represented as a scalar combination of $\vec{a}$ and $\vec{b}$.

b) To be specific we will henceforth fix

$$\vec{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers α and β such that

$$\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b}$$

c) Consider now also a third vector

$$\vec{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for (α, β, γ) such that

$$\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b} + \gamma\vec{c}$$

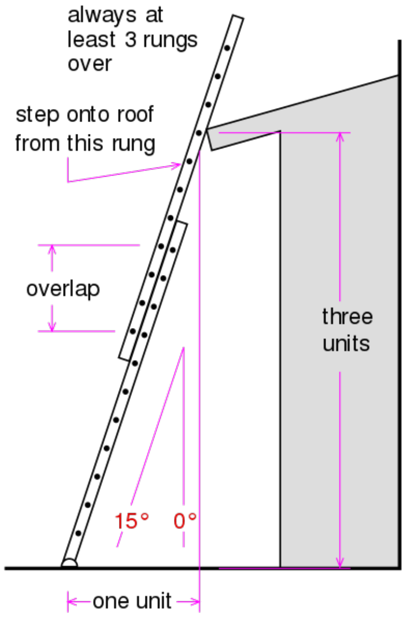
What is the general constraints on (α, β, γ) such that $\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b} + \gamma\vec{c}$. What does this imply on the number of solutions?

d) Discuss now the linear dependence of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ by exploring the solutions of

$$\vec{0} = \alpha\,\vec{a} + \beta\,\vec{b} + \gamma\vec{c}$$

How are the constraints for the null vector related to those obtained in part c)?

Problem 3.4. Forces acting on a ladder



Original: Bill Bradley – Vector: Sarang
[Public domain from wikimedia]

The sketch to the left shows the setup of a ladder leaning to the roof of a hut. The indicated angle from the downwards vertical to the ladder will be denoted as θ. There is a gravitational force of magnitude Mg acting of a ladder of mass M. At the point where it leans to the roof there is a normal force of magnitude F_r acting from the roof to the ladder. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude γF_f.

a) In principle there also is a friction force $\gamma_r F_r$ acting at the contact from the ladder to the roof. Why is it admissble to neglect this force?
Remark: There are at least two good arguments.

b) Determine the vertical and horizontal force balance for the ladder. Is there a unique solution?

c) The feet of the ladder start sliding when F_f exceeds the maximum static friction force γF_g. What does this condition entail for the angle θ?
Assume that $\gamma \simeq 0.3$ What does this imply for the critical angle θ_c.

d) Where does the mass of the ladder enter the discussion? Do you see why?

Problem 3.5. Different basis for polynomials

We consider the set of polynomials $\mathbb{P}_N$ of degree N with real coefficients p_n, $n \in \{0, \ldots, N\}$,

$$\mathbb{P}_N := \left\{ \vec{p} = \left(\sum_{k=0}^{N} p_n \, x^k \right) \quad \text{mit} \;\; p_n \in \mathbb{R}, n \in \{0, \ldots, N\} \right\}$$

a) Demonstrate that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is a vector space when one adopts the operations

$$\forall \;\; \vec{p} = \left(\sum_{k=0}^{N} p_n \, x^k \right) \in \mathbb{P}_N \,, \quad \vec{q} = \left(\sum_{k=0}^{N} q_n \, x^k \right) \in \mathbb{P}_N \,, \;\text{and}\;\; c \in \mathbb{R} \;:$$

$$\vec{p} + \vec{q} = \left(\sum_{k=0}^{N} (p_k + q_k) \, x^k \right) \qquad \text{and} \qquad c \cdot \vec{p} = \left(\sum_{k=0}^{N} (c \, p_k) \, x^k \right) \,.$$

(b) Demonstrate that

$$\vec{p} \cdot \vec{q} = \left(\int_0^1 \mathrm{d}x \left(\sum_{k=0}^{N} p_k \, x^k \right) \left(\sum_{j=0}^{N} q_j \, x^j \right) \right) \,,$$

establishes a scalar product on this vector space.

(c) Demonstrate that the three polynomials $\vec{b}_0 = (1)$, $\vec{b}_1 = (x)$ und $\vec{b}_2 = (x^2)$ form a basis of the vector space $\mathbb{P}_2$: For each polynomial $\vec{p}$ aus $\mathbb{P}_2$ there are real numbers x_k, $k \in \{0, 1, 2\}$, such that $\vec{p} = x_0 \, \vec{b}_0 + x_1 \, \vec{b}_1 + x_2 \, \vec{b}_2$. However, in general we have $x_i \neq \vec{p} \cdot \vec{b}_i$. Why is that?
Hint: Is this an orthonormal basis?

(d) Demonstrate that the three vectors $\hat{e}_0 = (1)$, $\hat{e}_1 = \sqrt{3}\,(2\,x - 1)$ and $\hat{e}_2 = \sqrt{5}\,(6\,x^2 - 6\,x + 1)$ are orthonormal.

(e) Demonstrate that every vector $\vec{p} \in \mathbb{P}_2$ can be written as a scalar combination of $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$,

$$\vec{p} = (\vec{p} \cdot \hat{e}_0) \, \hat{e}_0 + (\vec{p} \cdot \hat{e}_1) \, \hat{e}_1 + (\vec{p} \cdot \hat{e}_2) \, \hat{e}_2 \,.$$

Hence, $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$ form an orthonormal basis of $\mathbb{P}_2$.

*(f) Find a constant c and a vector $\hat{n}_1$, such that $\hat{n}_0 = (c\,x)$ and $\hat{n}_1$ form an orthonormal basis of $\mathbb{P}_1$.

Bonus Problem

Problem 3.6. Systems of linear equations

A system of N linear equations of M variables $x_1, \dots x_M$ comprises N equations of the form

$$\begin{aligned} b_1 &= a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1M}\,x_M \\ b_2 &= a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2M}\,x_M \\ \vdots &\quad \vdots \\ b_N &= a_{N1}\,x_1 + a_{N2}\,x_2 + \cdots + a_{NM}\,x_M \end{aligned}$$

where $b_i, a_{ij} \in \mathbb{R}$ for $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$.

a) Demonstrate that the linear equations $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$ form a vector space when one adopts the operations

$$\begin{aligned} \forall \quad & \vec{p} = \big[p_0 = p_1\,x_1 + p_2\,x_2 + \cdots + p_M\,x_M\big] \in \mathbb{L}_N\,, \\ & \vec{q} = \big[q_0 = q_1\,x_1 + q_2\,x_2 + \cdots + q_M\,x_M\big] \in \mathbb{L}_N\,, \\ & c \in \mathbb{R} \;: \end{aligned}$$

$$\begin{aligned} \vec{p} + \vec{q} &= \big[p_0 + q_0 = (p_1 + q_1)\,x_1 + (p_2 + q_2)\,x_2 + \cdots + (p_M + q_M)\,x_M\big] \\ c \cdot \vec{p} &= \big[c\,p_0 = c\,p_1\,x_1 + c\,p_2\,x_2 + \cdots + c\,p_M\,x_M\big]\,. \end{aligned}$$

How do these operations relate to the operations performed in Gauss elimination to solve the system of linear equations?

b) The system of linear equations can also be stated in the following form

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M$$

$$\vec{b} = x_1\,\vec{a}_1 + x_2\,\vec{a}_2 + \cdots + x_M\,\vec{a}_M$$

where $\vec{b}$ is expressed as a linear combination of $\vec{a}_1, \dots \vec{a}_M$ by means of the numbers $x_1, \dots, x_M$. What do the conditions on linear independence and representation of vectors by means of a basis tell about the existence and uniqueness of the solutions of a system of linear equations.