Homework Exercises 2

Your solution to the problems 2.4–2.6 should be handed in/presented
either during a seminar on Monday, Oct 28, at 11:00,
or in my mail box at ITP, room 105b, by Monday, Oct 28, 13:00.
Please submit only Problem 2.6 when you participate in the seminar.

Warm-up

Problem 2.1. Brackets are crucial

In the absence of brackets logical operators are evaluated in the order negation $\neg$, and $\wedge$, or $\vee$, implication $\Rightarrow$, and equivalence $\Leftrightarrow$. Changing the order, e.g. by adding brackets, can severely change the truth value of a statement. Explore this by comparing three different ways to put brackets into one of the following formulas

a) $A \wedge B \vee C \Leftrightarrow B \wedge C \Rightarrow A$ b) $A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$

To clarify my intention I provide an example for two possibilities

A	B	C	$A \wedge (B \vee C) \Leftrightarrow B \wedge C \Rightarrow A$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

A	B	C	$A \wedge (B \vee C) \Leftrightarrow B \wedge (C \Rightarrow A)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

Problem 2.2. Sets and number theory

Let $T(a)$ be the set of all divisors of a number $a \in \mathbb{N}$.
For instance, $T(6) = \{1, 2, 3, 6\}$, $T(7) = \{1, 7\}$, and $T(8) = \{1, 2, 4, 8\}$.
What is the relation between the two sets $T(a) \cup T(b)$ and $T(a \cdot b)$ for numbers $a, b \in \mathbb{N}$?

Problem 2.3. Intervals of real numbers

Describe the following subsets of $\mathbb{R}$ as unions of disjoint intervals[1]

a) $[-1, 4] \backslash [1, 2[$ b) $[2, 4) \cup ([3, 10] \backslash (]3, 4[\cup [6, 7]))$

[1] We denote an open set here with square bracket that open in the "wrong" direction, rather than with brackets. This will be done whenever this notion is more transparent because the brackets (and) are needed for different purposes.

Homework Problems

Problem 2.4. Relations between sets

Let A, B, C, and D be pairwise distinct elements. Select one of the symbols

$$\in, \quad \notin, \quad \ni, \quad \not\ni, \quad \subset, \quad \not\subset, \quad \supset, \quad \not\supset, \quad =$$

in place of the box $\square$ in order to have true statements.

a) $\{A, B\} \ \square \ \{A, B, C\}$,

b) $\{A\} \ \square \ B$,

c) $\{\emptyset\} \ \square \ \emptyset$,

d) $\{\{A\}\} \ \square \ \{\{A\}, \{B\}\}$,

e) $A \ \square \ \{A, B, C\}$,

f) $\{A, C, D\} \cap \{A, B\} \ \square \ \{A, B, C, D\}$,

g) $\{A, C, D\} \setminus \{A, B\} \ \square \ \{A, B, C\}$,

h) $\{A, C, D\} \cup \{A, B\} \ \square \ A$.

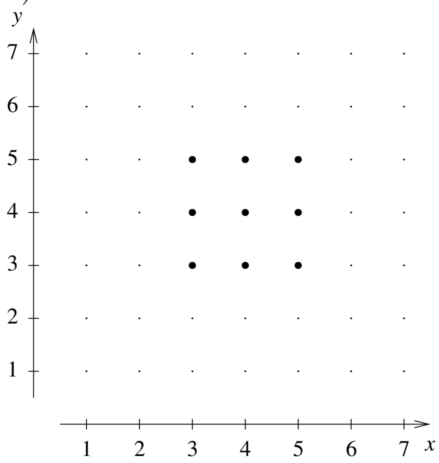
Problem 2.5. Challenges in drawing circles

In the realm of real numbers the set $K = \{(x, y) \in \mathbb{R} \mid (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$ describes a filled circle. Now we consider the set $N = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq x, y \leq 7\}$. Which of the following subsets of N can be specified as a "circle" in the form

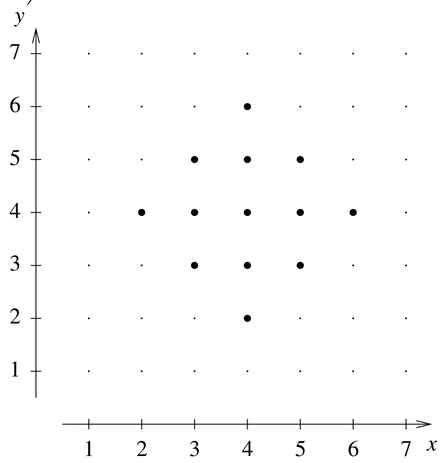
$$K_r = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x - 4)^2 + (y - 4)^2 \leq r^2\}$$

Which values of r are admissible if this is possible?

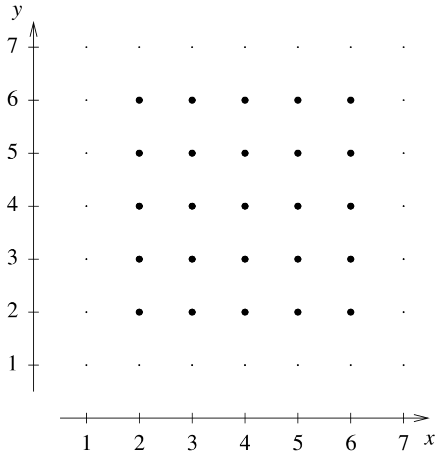
a)



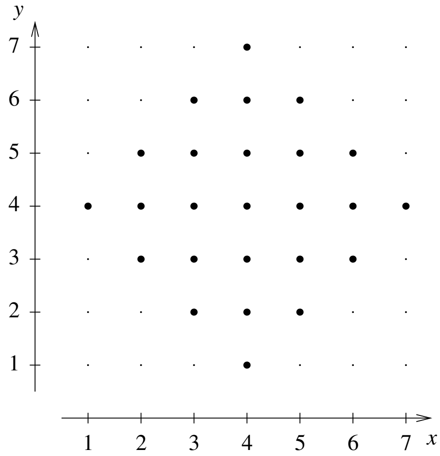
b)



c)



d)



Problem 2.6. Dihedral group D_8

In the figure below we demonstrate the action of the Dihedral group D_8 by its action on a stop sign. To this end we show the result of applying any of its elements to the configuration in the top left.

a) How many mirror axes and how many distinct rotation angles will transform the octagon in itself?

b) Find a rotation d and a reflection s that allow you to generate each element of D_8 by repeated action of d and s. Specify how they should be applied to generate the 16 elements of D_8.

Bonus. Note: Some choices do not work. Can you find an example?

c) How many times do you have to apply d to generate the neutral element of D_8? How many times do you have to apply s?

d) How can you represent sds as composition of only d?

e) What is still to be done to proof that D_8 is a group?

Bonus Problem

Problem 2.7. A group where the operation is a cross sum
We explore the set $M = \{1, 2, 4, 5, 7, 8\}$, and combine the elements by multiplication and subsequently taking the cross sum. For example $3 \circ 5 = 6$, because the product of 3 and 5 is $3 \cdot 5 = 15$, and the cross sum of 15 is $1 + 5 = 6$. For larger numbers we repeatedly take the sum of the digits until we arrive at a single-digit number. For instance, $7 \circ 8 = 2$, because $7 \cdot 8 = 56$ with cross sum $5 + 6 = 11$, and eventually we obtain $1 + 1 = 2$. We will show now that $(M, \circ)$ is a group, and discuss its relation to the dihedral group D_3.

a) Verify that the operation $\circ$ on M is commutative.

b) Fill in the Caley table for the operation $\circ$ on M. How do you see in this table that the operation is commutative?

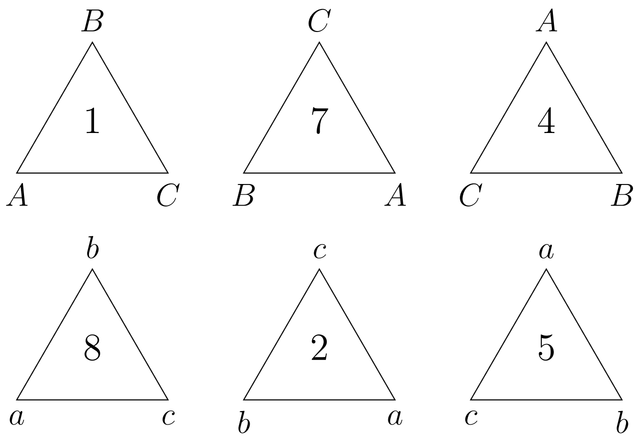
$\circ$	1	2	4	5	7	8
1						
2						
4						
5						
7						
8						

c) Verify that the set M is closed with respect to the operation $\circ$. Besides by checking the Caley table, this can also be shown by a direct calculation. How? **Hint:** Observe the divisibility rule for 3 and 9.

d) What is the neutral element of this group?

e) What are the neutral elements for the other elements of M?

f) We arrange the elements graphically as follows:



Demonstrate that the element 7 amounts to a 120° rotation of the triangles. Henceforth, we denote it as d.

Verify that the element 8 swaps between small and capital letters at the vertices. Henceforth, we denote it as s.

g) How can the elements of the group be expressed in terms of s and d?

h) Use the commutatively of the operation and the representation of the group elements obtained in (g) to proof associativity.