Lecture Notes by Jürgen Vollmer

Theoretical Mechanics

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Fakultät für Physik und Geowissenschaften, Universität Leipzig

LECTURES DELIVERED AT FAKULTÄT FÜR PHYSIK UND GEOWISSENSCHAFTEN, UNIVERSITÄT LEIPZIG
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Die Philosophie steht in diesem großen Buch geschrieben, dem Universum, das unserem Blick ständig offen liegt. Aber das Buch ist nicht zu verstehen, wenn man nicht zuvor die Sprache erlernt und sich mit den Buchstaben vetraut gemacht hat, in denen es geschrieben ist. Es ist in der Sprache der Mathematik geschrieben, und deren Buchstaben sind Kreise, Dreiecke und andere geometrische Figuren, ohne die es dem Menschen unmöglich ist, ein einziges Wort davon zu verstehen; ohne diese irrt man in einem dunklen Labyrinth herum.

GALILEO GALILEI, Il Saggiatore, 1623

Die Mathematik ist das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten: sie baut die verbindende Brücke und gestaltet sie immer tragfähiger. Daher kommt es, daß unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Natur beruht, ihre Grundlage in der Mathematik findet.

DAVID HILBERT, Ansprache "'Naturerkennen und Logik"' am 8.9.1930 während des Kongresses der Vereinigung deutscher Naturwissenschafter und Mediziner

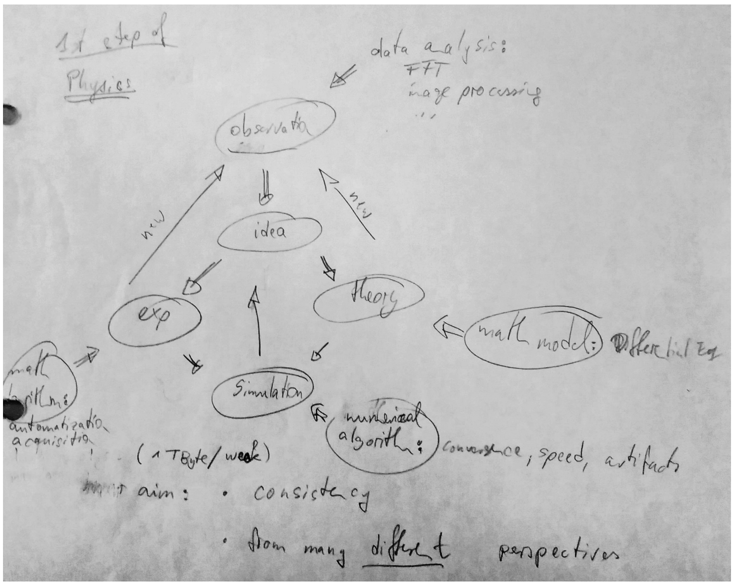
Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.

ALBERT EINSTEIN Festvortrag "'Geometrie und Erfahrung"' am 27.1.1921 vor der Preußischen Akademie der Wissenschaften

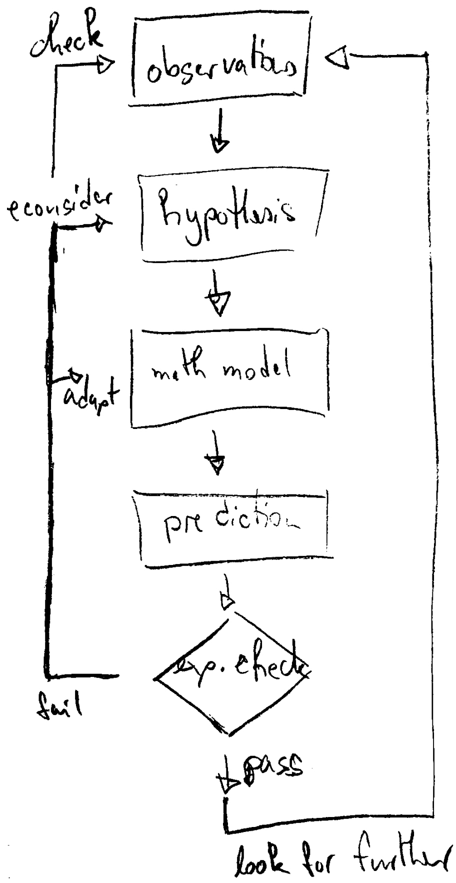
Preface

Die ganzen Zahlen hat der liebe Gott geschaffen,
alles andere ist Menschenwerk.
Leopold Kronecker

Almost 400 years ago Galilei Galileo expressed the credo of modern sciences: The language of mathematics is the appropriate instrument to decode the secrets of the universe. Arguably the fruits of this enterprise are more visible today than they have ever been in the past. Mathematical models are the cornerstone of modern science and engineering. They provide the tools for optimizing engines, and the technology for data and communication sciences. No car will run, no plane will fly, no cell phone ring without the technical equipment and the software to make it run. Moreover, again and again the challenges of physics models inspired the development of new mathematics. Indeed, physics and mathematics take complementary perspectives: Mathematicians strive for a logically stringent representation of the structure of theories and models. Physicists adopt mathematics as a tool to speak about and better understand nature:



The present Lecture Notes are developed to accompany courses on “Theoretical Mechanics” for physics freshmen in the international physics program and for students in the teacher education program of the Universität Leipzig. The course addresses mechanics problems to introduce the students to concepts and strategies aiming at a quantitative description of observations.



To meet that aim the lectures strive to meet several purposes:

a) They introduce the concept of a mathematical model, its predictions, and how they relate to observations.

b) They present strategies adopted to develop a model, to explore its predictions, to falsify models, and to refine them based on comparison to observations.

c) They introduce mathematical concepts used in this enterprise: dimensional analysis, non-dimensionalization, complex numbers, vector calculus, and ordinary differential equations.

d) They provide an introduction to Newtonian and Lagrangian Mechanics.

Our approach to mathematical concepts is strongly biased to developing skills to apply mathematical tools in a modeling context, rather than striving for mathematical rigor. For the latter we point out potential pitfalls based on physical examples, and refer the students to maths classes.

The material is organized in chapters that address subsequent mathematical and physical topics. Each chapter is introduced by a

add reading plan

physics illustration problem. Then, we develop and discuss relevant new concepts. Subsequently, we provide a worked examples. One of them will be the solution of the problem sketched in the introduction. Finally, there is a section with different types of problems:

a. quickies to test conceptual understanding and highlight the new concepts.

b. exercises to gain practice in employing the concepts.

c. more elaborate exercises where the new concepts are used to discuss non-trivial problems.

d. exercises that provide complementary insight based on Python and Sage programs

add more explanation

e. teasers with challenging problems. Typically these exercises require a non-trivial combination of different concepts that have been introduced in earlier chapters.

At the end of the chapter we recommend additional literature and provide an outlook for further reading.

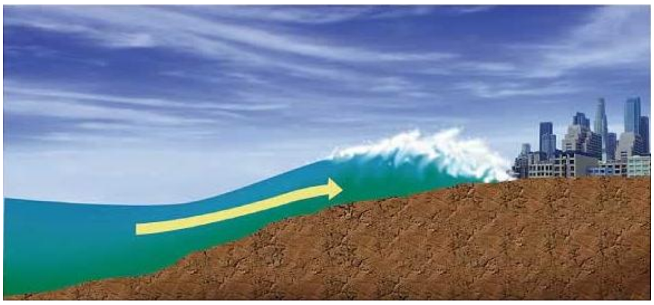
I am grateful to Robin Barta, Kolya Lettl, Menna Noufal and Maurice Zeuner for feedback on these notes, and some help with typing.

acknowledge further co-workers

I am eager to receive feedback. It is crucial for the development of this project to learn about typos, inconsistencies, confusing or incomplete explanations, and suggestions for additional material (contents as well as links to papers, books and internet resources) that should be added in forthcoming revisions. Everybody who is willing to provide feedback will be invited to a coffee in Café Corso.

Contents

1
Basic Principles



-Ilhador-/wikimedia, Public domain

At the end of this chapter we will be able to estimate the speed of a Tsunami wave.

1.1 Basic Notions of Mechanics

Definition 1.1: System

A mechanical *system* is comprised of particles labeled by an index $i \in \mathbb{I}$, that have masses m_i, reside at the positions $\boldsymbol{x}_i$, and move with velocities $\boldsymbol{v}_i$.

Remark 1.1. We say that the system has N particles when $\mathbb{I} = \{1, \ldots, N\}$.

Remark 1.2. The arrows indicate here that $\boldsymbol{x}_i$ describes a position in space. For a D-dimensional space one needs D numbers[1] to specify the position, and $\boldsymbol{x}_i$ may be thought of as a vector in $\mathbb{R}^D$. We say that $\boldsymbol{x}_i$ is a D-vector.

[1] Strictly speaking we do not only need numbers, but must also indicate the adopted units.

Remark 1.3. In order to emphasize the close connection between positions and velocities, the latter will also be denoted as $\dot{\boldsymbol{x}}$.

Example 1.1: A piece of chalk

We wish to follow the trajectory of a piece of chalk through the lecture hall. In order to follow its position and orientation in space, we *decide* to model it as a set of two masses that are localized at the tip and at the tail of the chalk. The positions of these two masses $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ will both be vectors in $\mathbb{R}^3$. For instance we can indicate the shortest distance to three walls that meet in one corner of the lecture hall. In this model we have $N = 2$ and $D = 3$.

Definition 1.2: Degrees of Freedom (DOF)

A system with N particles whose positions are described by D-vectors has $D\,N$ *degrees of freedom (DOF)*.

Remark 1.4. Note that according to this definition the number of DOF is a property of the model. For instance, the model for the piece of chalk has $D\,N = 6$ DOF. However, the length of the piece of chalk does not change. Therefore, one can find an alternative description that will only evolve 5 DOF. (We will come back to this in due time.)

Definition 1.3: State Vector

The position of all particles can be written in a single *state vector*, $\boldsymbol{q}$, that specifies the positions of all particles. Its components are called coordinates.

Remark 1.5. For a system with N particles whose positions are specified by D-dimensional vectors, $\boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,D})$, the vector $\boldsymbol{q}$ takes the form $\boldsymbol{q} = (x_{1,1}, \ldots, x_{1,D}, x_{2,1}, \ldots, x_{2,D}, \ldots, x_{N,1}, \ldots, x_{N,D})$, which comprises the coordinates $x_{1,1}, \ldots, x_{N,D})$. For conciseness we will also write $\boldsymbol{q} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. The vector $\boldsymbol{q}$ has DOF number of entries, and hence $\boldsymbol{q} \in \mathbb{R}^{DN}$.

Remark 1.6. The velocity associated to $\boldsymbol{q}$ will be denoted as $\dot{\boldsymbol{q}} = (\dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N)$.

Definition 1.4: Phase Vector

The position and velocities of all particles form the *phase vector*, $\boldsymbol{\Gamma} = (\boldsymbol{q}, \dot{\boldsymbol{q}})$.

Definition 1.5: Trajectory

The *trajectory* of a system is described by specifying the time dependent functions

$$\begin{aligned} & \boldsymbol{x}_i(t), \boldsymbol{v}_i(t), \quad i = 1, \ldots, N \\ \text{or} \quad & \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t) \\ \text{or} \quad & \boldsymbol{\Gamma}(t) \end{aligned}$$

Definition 1.6: Initial Conditions (IC)

For $t \in [t_0, \infty)$ the trajectory is uniquely determined by its *initial conditions (IC)* for the positions $\boldsymbol{x}_i(t_0)$ and velocities $\boldsymbol{v}_i(t_0)$, i.e. the point $\boldsymbol{\Gamma}(t_0)$ in phase space.

Remark 1.7. This definition expresses that the future evolution of a system is *uniquely* determined by its ICs. Such a system is called deterministic. Mechanics addresses the evolution of deterministic systems. At some point in your studies you might encounter stochastic dynamics where different rules apply.

Example 1.2: Throwing a javelin

The ICs for the flight of a javelin specify where it is released, x_0, when it is thrown, the velocity v_0 at that point of time, and the orientation of the javelin. In a good trial the initial orientation of the javelin is parallel to its initial velocity v_0, as shown in Figure 1.1

based on Atalanta, creativecommons, CC BY-SA 3.0
Figure 1.1: Initial conditions for throwing a javelin, cf. Example 1.2.

Remark 1.8. In repeated experiments the ICs will be (slightly) different, and one observes different trajectories.

1. A seasoned soccer player will hit the goal in repeated kicks. However, even a professional may miss occasionally.

2. A bicycle involves a lot of mechanical pieces that work together to provide a predictable riding experience.

3. A lottery machine involves a smaller set of pieces than a bike, but it is constructed such that unnoticeably small differences of initial conditions give rise to noticeably different outcomes. The outcome of the lottery can not be predicted, in spite of best efforts to select identical initial conditions.

Definition 1.7: Constant of Motion

A function of the positions x_i and velocities v_i is called a *constant of motion*, when it does not evolve in time.

Remark 1.9. For a given initial conditions a constant of motion takes the same value for the full trajectory. However, it may take different values for different trajectories, i.e. different choices of initial conditions.

Example 1.3: Length of a piece of chalk

During the flight the positions x_1 and x_2 of the piece of chalk will change. However, the length L of the piece of chalk will not, and at any given time it can be determined from x_1 and x_2. Hence, L is a constant of motion that takes the same value for all trajectories of the piece of chalk.

Example 1.4: Energy conservation for the piece of chalk

We will see that the sum of the potential and the kinetic energy is conserved during the flight of the piece of chalk. This sum, the total energy E, is a constant of motion. The potential energy depends on the position and the kinetic energy is a function of the velocity. Trajectories that start at the same position with different speed will therefore have different total energy. Hence, E is a constant of motion that can take different value for different trajectories of the piece of chalk.

Definition 1.8: Parameter

In addition to the ICs the trajectories will depend on *parameters* of the system. Their values are fixed for a given system.

Example 1.5: A piece of chalk

For the piece of chalk the trajectory will depend on whether the hall is the Theory Lecture Hall in Leipzig, a briefing room in a ship during a heavy storm, or the experimental hall of the ISS space station. To the very least one must specify how the gravitational acceleration acts on the piece of chalk, and how the room moves in space.

Remark 1.10. The set of parameters that appear in a model depends on the *choices* that one makes upon setting up the experiment. For instance

Beckham's banana kicks can only be understood when one accounts for the impact of air friction on the soccer ball.

Air friction will not impact the trajectory of a small piece of talk that I through into the dust bin.

By adopting a clever choice of the parameterization the trajectory of the piece of chalk can be described in a setting with 5 DOF. The length of the piece of chalk will appear as a parameter in that description.

Definition 1.9: Physical Quantities

Positions, velocities and parameters are *physical quantities* that are characterized by at least one number and a unit.

Example 1.6: Physical Quantities

1. The mass of a soccer ball can be fully characterized by a number and the unit kilogram (kg), e.g. $M \approx 0.8\,\mathrm{kg}$.
2. The length of a piece of chalk can be fully characterized by a number and the unit meter (m), e.g. $L \approx 7 \times 10^{-2}\,\mathrm{m}$.
3. The length T of a year can be characterized by a number and the unit second, e.g. $T \approx \pi \times 10^7\,\mathrm{s}$.
4. The speed of a car can be fully characterized by a number and the unit, e.g. $v \approx 42\,\mathrm{km\,h^{-1}}$.
5. A position in a D-dimensional space can fully be characterized by D numbers and the unit meter.
6. The velocity of a piece of chalk flying through the lecture hall can be characterized by three numbers and the unit m/s. However, one is missing information in that case

about its rotation.

Remark 1.11. Analyzing the units of the parameters of a system provides a fast way to explore and write down functional dependencies. When doing so, the units of a physical quantity Q are denoted by $[Q]$. For instance for the length L of the piece of chalk, we have $[L] = \mathrm{m}$. For a dimensionless quantity d we write $[d] = 1$.

Example 1.7: Changing units

Suppose we wish to change units from km/h to m/s. A transparent way to do this for the speed of the car in the example above is by multiplications with one

$$v = 72 \frac{\mathrm{km}}{\mathrm{h}} \, \frac{1\,\mathrm{h}}{3.6 \times 10^3\,\mathrm{s}} \, \frac{1 \times 10^3\,\mathrm{m}}{1\,\mathrm{km}} = \frac{72}{3.6} \mathrm{m\,s^{-1}} = 20\,\mathrm{m\,s^{-1}}$$

Definition 1.10: Dynamics

The characterization of all possible trajectories for all admissible ICs is called *dynamics* of a system.

1.2 Dimensional Analysis

Mathematics does not know units. Experimental physicists hate large sets of parameters because the sampling of high-dimensional parameter space is tiresome. A remedy to both issues is offered by the Buckingham-Pi-Theorem. We state it here in a form accessible with our present level of mathematical refinement. The discussion of a more advanced formulation may appear as a homework problem later on on this course.

Theorem 1.1: Buckingham-Pi-Theorem

A dynamics with n parameters, where the positions $\boldsymbol{q}$ and the parameters involve the three units meter, seconds and kilogram, can be rewritten in terms of a *dimensionless dynamics* with $n-3$ parameters, where the positions $\boldsymbol{\xi}$, velocities $\boldsymbol{\zeta}$, and parameters π_j with $j \in \{1, \dots, n-1\}$ are given solely by numbers.

Example 1.8: Non-dimensionalization for a pendulum

Let $\boldsymbol{x}$ denote the position of a pendulum of mass M that is attached to a chord of length L and swinging in a gravitational field $\boldsymbol{g}$ of strength g (see Figure 1.2).
The units of these quantities are $[x] = \mathrm{m}$, $[M] = \mathrm{kg}$, $L = \mathrm{m}$, and $[g] = \mathrm{m/s^2}$, respectively. There are three parameters, $n = 3$, plus the direction of $\boldsymbol{g}$.
In this problem we choose L as length scale and $\sqrt{L/g}$

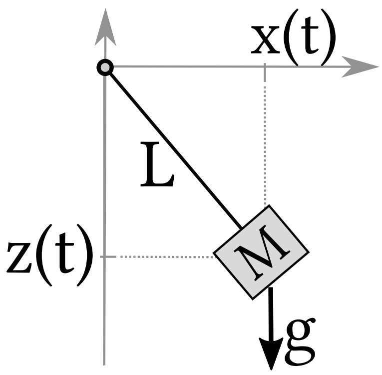


Figure 1.2: Pendulum discussed in Example 1.8

as time scale. Then the dimensionless positions will be $\boldsymbol{\xi} = \boldsymbol{x}/L$, the dimensionless velocities will be $\boldsymbol{\zeta} = \dot{\boldsymbol{x}}/\sqrt{g\,L}$. There is no way to turn M into a dimensionless parameter. Therefore the evolution of (ξ, ζ) can not depend on M. The only dimensionless parameter that remains in the model is the direction of $\boldsymbol{g}$.

Example 1.9: Non-dimensionalization for the flight of a piece of chalk

Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote the position of the tip and the tail of a model for a piece of chalk, where tip and tail are associated to masses m_1 and m_2. The piece of chalk has a length L. It performs a free flight in a gravitational field with acceleration $\boldsymbol{g}$ of strength g.
The units of these quantities are $[\boldsymbol{x}_i] = \mathrm{m}$, $[m_i] = \mathrm{kg}$, $L = \mathrm{m}$, and $[g] = \mathrm{m/s^2}$, respectively. There are four parameters, $n = 4$, plus the direction of $\boldsymbol{g}$.
In this problem we choose L as length scale and $\sqrt{L/g}$ as time scale. Then the dimensionless positions will be $\boldsymbol{\xi}_i = \boldsymbol{x}_i/L$, the dimensionless velocities will be $\boldsymbol{\zeta} = \dot{\boldsymbol{x}}_i/\sqrt{g\,L}$. The two masses m_1 and m_2 give rise to the dimensionless parameter $\pi_1 = m_1/m_2$, and in three dimensions the direction of $\boldsymbol{g}$ must be characterized by another two dimensionless parameters.

Proof of the Buckingham-Pi-Theorem. We first look for combinations of the parameters with the following units

$$\mathrm{m} = \left[p_1^{\alpha_1}\right]\,\left[p_2^{\alpha_2}\right]\,\dots\,\left[p_n^{\alpha_n}\right]$$
$$\mathrm{s} = \left[p_1^{\beta_1}\right]\,\left[p_2^{\beta_2}\right]\,\dots\,\left[p_n^{\beta_n}\right]$$
$$\mathrm{kg} = \left[p_1^{\gamma_1}\right]\,\left[p_2^{\gamma_2}\right]\,\dots\,\left[p_n^{\gamma_n}\right]$$

Each of these equations involves constraints on the exponents in order to match the exponents of the three units that can be expressed as a system of linear equations. The solvability conditions for such systems imply that they conditions can always be met by an appropriately chosen set of three parameters. Without loss of generality we denote them as p_1, p_2 and p_3, and we have

$$\begin{aligned} \mathrm{m} &= \left[p_1^{\alpha_1}\right]\,\left[p_2^{\alpha_2}\right]\,\left[p_3^{\alpha_n}\right] \\ \mathrm{s} &= \left[p_1^{\beta_1}\right]\,\left[p_2^{\beta_2}\right]\,\left[p_3^{\beta_n}\right] \\ \mathrm{kg} &= \left[p_1^{\gamma_1}\right]\,\left[p_2^{\gamma_2}\right]\,\left[p_3^{\gamma_n}\right] \end{aligned} \tag{1.2.1}$$

Thus we use the parameters $p_1, \dots, p_3$ to remove the units from our description. In its dimensionless form it will involve the positions and velocities

$$\xi = q\, p_1^{-\alpha_1}\, p_2^{-\alpha_2}\, p_3^{-\alpha_n}$$

$$\zeta = \dot{q}\, p_1^{\beta_1-\alpha_1}\, p_2^{\beta_2-\alpha_2}\, p_3^{\beta_n-\alpha_n}$$

Similarly, the dimensionless form of the parameters p_i of the dynamics are obtained by multiplying the original parameters with appropriate powers of the expressions (1.2.1) of the units. For p_1 to p_3 this gives rise to one. Additional parameters will turn into dimensionless groups of parameters that provide π_1 to π_{n-3}. □

1.2.1 Self Test

Problem 1.1. **Oscillation Period of a Particle attached to a spring** In a gravitational field with acceleration $g_{\mathrm{Moon}} = 1.6\,\mathrm{m/s^2}$ a particle of mass $M = 100\,\mathrm{g}$ is hanging at a spring with spring constant $k = 1.6\,\mathrm{kg/s^2}$. It oscillates with period T when it is slightly pulled downwards and released. We describe the oscillation by the distance $x(t)$ from its rest position.

a) Determine the dimensionless distance $\xi(t)$, and the associated dimensionless velocity $\zeta(t)$.

b) Provide an order-of-estimate guess of the oscillation period T.

Problem 1.2. **Useful numbers and unit conversions.**

a) Verify that
   - one nano-century amounts to π seconds,
   - a colloquium talk at our Physics Department must not run take longer than a micro-century,
   - a generous thumb-width amounts to one atto-parsec.

b) The Physics Handbook of Nordling and Österman (2006) defines a beard-second, i. e. the length an average beard grows in one second, as 10 nm. In contrast, Google Calculator uses a value of only 5 nm. I prefer the one where the synodic period of the moon amounts to a beard-inch. Which one will that be?

c) In the furlong–firkin–fortnight (FFF) unit system one furlong per fortnight amounts to the speed of a tardy snail (1 centimetre per minute to a very good approximation), and one micro-fortnight was used as a delay for user input by some old-fashinoed computers (it is equal to 1.2096 s). Use this information to determine the length of one furlong.

Problem 1.3. **Earth orbit around the sun**

a) Light travels with a speed of $c \approx 3 \times 10^8\,\mathrm{m\,s^{-1}}$, and it takes 500 s to travel from Sun to Earth. What is the Earth-Sun distance D, i. e. one Astronomical Unit (AU) in meters?

b) The period of the trajectory of the Earth around the Sun depends on D, on the mass $M = 2 \times 10^{30}$ kg of the sun, and on the gravitational constant $G = 6.7 \times 10^{-11}\,\mathrm{m}^3/\mathrm{kg\,s}^2$. Estimate, based on this information, how long it takes for the Earth to travel once around the sun.

c) Express your estimate in terms of years. The estimate of (b) is of order one, but still off by a considerable factor. Do you recognize the numerical value of this factor?

d) Upon discussing the trajectory $\boldsymbol{x}(t)$ of planets around the sun later on in this course, we will introduce dimensionless positions of the planets $\boldsymbol{\xi}(t) = \boldsymbol{x}(t)/L = (x_1(t)/L, x_2(t)/L, x_3(t)/L)$. How would you define the associated dimensionless velocities?

1.3 *Order-of-magnitude guesses*

Many physical quantities take a value close to one when they are expressed in their “natural” dimensionless units. When the choice is unique, then clearly it is also natural. Otherwise, the appropriate choice is a matter of experience.

We will come back to this when we employ non-dimensionalization in the forthcoming discussion. We demonstrate this based on a discussion of

Example 1.10: The period of a pendulum

We consider a pendulum of mass M attached at a stiff bar of negligible mass. With this bar it is fixed to a pivot at a distance L from the mass such that it can swing in a gravitational field inducing an acceleration g. In this example we make use of the fact that the bar has fixed length L, and describe the position of the mass by the angel $\theta(t)$ (see Figure 1.3).

As discussed in Example 1.8 the dimensionless time unit for this problem is $\sqrt{L/g}$. Hence we estimate that the period T of the pendulum is of the order of $T \simeq \sqrt{L/g}$. Explicit calculations to be performed later on will reveal that this estimate is off by a factor 2π when the amplitude is small, $|\theta(t)| \ll 1$. For large oscillation amplitudes θ_0 the period will increase further, tending to infinity when θ_0 approaches π. Hence, we conclude that

$$T = f(\theta_0)\sqrt{L/g} \qquad \text{with} \;\; f(\theta_0) \simeq 2\pi \;\; \text{for} \;\; \theta_0 \ll 1\,.$$

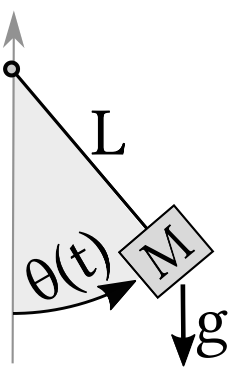


Figure 1.3: Pendulum discussed in Example 1.10

Example 1.11: The speed of Tsunami waves

A Tsunami wave is a water wave that is generated by an earth quake or an underwater land slide. Typical wave lengths are of an order of magnitude $\lambda = 100\,\text{km}$. They travel through the ocean that has an average depth of about $D = 4\,\text{km}$, much smaller than λ. Therefore, we expect that the wave speed v_{Tsunami} is predominantly set by the ocean depth and the gravitational acceleration $g \approx 10\,\text{m/s}^2$, i.e.

$$v_{\text{Tsunami}} \approx \sqrt{gD} = 2 \times 10^2\,\text{m/s} \approx 700\,\text{km/h}$$

This estimate suggests that the 2004 Indian Ocean Tsunami traversed the distance from Indonesia to the East African coast, $L \approx 10\,000\,\text{km}$, in about

$$\frac{L}{v_{\text{Tsunami}}} \approx \frac{1 \times 10^4\,\text{km}}{700\,\text{km/h}} = \frac{100}{7}\text{h} \approx 15\,\text{h}$$

This is very close to the value of 16 h reported in Wikipedia.

Example 1.12: The period of Tsunami waves

In spite of their speed and devastating power, Tsunamis are very hard to detect on the open sea because their period T is very long. It can be estimated as the time that the wave needs to run once through its wavelength[2]

$$T \approx \frac{\lambda}{v_{\text{Tsunami}}} = \frac{\lambda}{\sqrt{gD}} = \frac{100\,\text{km}}{700\,\text{km/h}} = \frac{1}{7\,\text{h}} \approx 10\,\text{min}$$

Here, our estimate is too small by about a factor of three.

[2] Observe that this physical argument goes beyond the blind use of dimensional analysis. The equation for T involves the length scales λ and D in a non-trivial combination that is set by a physical argument.

We conclude that estimates based on dimensional analysis provide valuable insight in time scales of physical processes, even in situations where a detailed mathematical treatment is very delicate.

1.3.1 *Self Test*

Problem 1.4. **Printing the output of Phantom cameras**
With a set of three phantom cameras one can simultaneously follow the motion of 100 particles in a violent 3d turbulent flow. Data analysis of the images provides particle positions with a resolution of 25,000 frames per second. You follow the evolution for 20 minute, print it double paged with 8 coordinates per line and 70 lines per page. A bookbinder makes 12 cm thick books from every 1000 pages. You put these books into bookshelves with seven boards in each shelf. How many meters of bookshelves will you need to store your data on paper?

1.4 Problems

Problem 1.5. **Water Waves**
The speed of waves on the ocean depends only on their wave length L and the gravitational acceleration $g \simeq 10\,\mathrm{m/s^2}$.

a) How does the speed of the waves depend on L and g?

b) Unless it is surfing, the speed of a yacht is limited by its hull speed, i.e. the speed of a wave with wave length identical to the length of the yacht. Estimate the top speed of a 30 ft yacht.

c) Close to the beach the water depth H become a more important parameter than the wave length. How does the speed of the crest and the trough of the wave differ? What does this imply about the form of the wave?

1.5 Further Reading

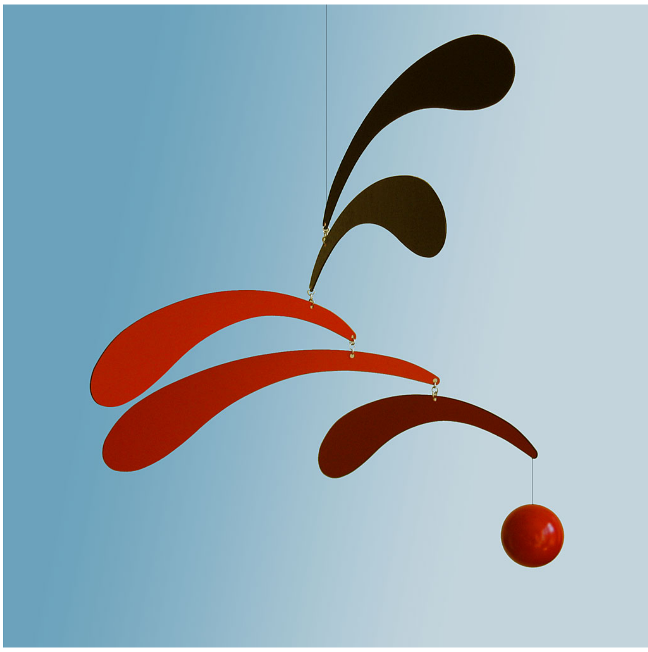
The first chapter of Großmann (2012) provides a clear and concise introduction to basic calculus with an emphasis on applications to physics problems.

Morin (2014, 2007) provides an excellent introduction to problem solving strategies in physics and dimensional analysis.

2

Balancing Forces and Torques

In Chapter 1 we observed that positions and velocities of particles are specified by indicating their unit, magnitude and directions. Hence, they are vectors. In the present chapter we learn how vectors are defined in mathematics, and how they are used and handled in physics. In order to provide a formal definition we introduce a number of mathematical concepts, like groups, that will be revisited in forthcoming chapters. As first important application we deal with balancing forces and torques.



Mobile (sculpture) in the style of Alexander Calder
Andrew Dunn / wikimedia CC BY-SA 2.0

At the end of this chapter we will be able to determine how a mobile hangs from the ceiling.

2.1 *Motivation and Outline: What is a Vector?*

In mechanics we use vectors to describe forces, displacements and velocities. A displacement describes the relative position of two points in space, and the velocity can be thought of as a distance divided by the time needed to go from the initial to the final point. (A mathematically more thorough definition will be given in Chapter 3.) For forces it is of paramount importance to indicate in which direction they are acting. Similarly, in contrast to speed, a velocity can not be specified in terms of a number with a unit, e.g. $5\,\mathrm{m/s}$. By its very definition one also has to specify the direction of motion. Finally, also a displacement involves a length specification and a direction.

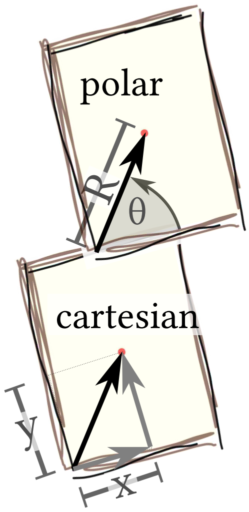


Figure 2.1: The displacement of the red point from the bottom left corner to the the middle of the page can either be specified by the direction θ and the distance R (polar coordinates, top), or by the distances x and y along the sides of the paper (Cartesian coordinates, bottom).

add more explanation

Example 2.1: Displacement of a red dot from the lower left corner to the middle of a paper

This displacement is illustrated in Figure 2.1. It can either be specified in terms of the distance R of the point from the corner and the angle θ of the line connecting the points and the lower edge of the paper (i.e. the direction of the point). Alternatively, it can be given in terms of two distances (x, y) that refer to the length x of a displacement along the edge of the paper and a displacement y in the direction vertical to the edge towards the paper. This can be viewed as result of two subsequent displacements indicated by gray arrows.

In three dimensions, one has to adopt a third direction out of the plane used for the paper, and hence three numbers, to specify a displacements—or indeed any other vector.

	displacement $\boldsymbol{x} = (x_1, x_2, x_3)$	velocity $\boldsymbol{v} = (v_1, v_2, v_3)$	force $\boldsymbol{F} = (f_1, f_2, f_3)$
unit	$[\boldsymbol{x}] = \mathrm{m}$	$[\boldsymbol{v}] = \mathrm{m\,s^{-1}}$	$[\boldsymbol{F}] = \mathrm{kg\ m/s^2}$
magnitude	$\lvert\boldsymbol{x}\rvert = \sqrt{x_1^2 + x_2^2 + x_3^2}$	$\lvert\boldsymbol{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	$\lvert\boldsymbol{F}\rvert = \sqrt{f_1^2 + f_2^2 + f_3^2}$
direction	$\hat{\boldsymbol{x}} = \boldsymbol{x}/\lvert\boldsymbol{x}\rvert$	$\hat{\boldsymbol{v}} = \boldsymbol{v}/\lvert\boldsymbol{v}\rvert$	$\hat{\boldsymbol{F}} = \boldsymbol{F}/\lvert\boldsymbol{F}\rvert$

A basic introduction of mechanics can be given based on this heuristic account of vectors. However, for the thorough exposition that serve as a foundation of theoretical physics a more profound mathematical understanding of vectors is crucial. Hence, a large part of this chapter will be devoted to mathematical concepts.

Outline

In the first part of this chapter we introduce the mathematical notions of sets and groups that are needed to provide a mathematically sound definition of a vector space. Sets are the most fundamental structure of mathematics. It denotes a collection of elements, e.g., numbers like the digits of our number system $\{1, 2, \ldots, 9\}$ or the set of students in my class. Mathematical structures refer to sets where the elements obey certain additional properties, like in groups and vector spaces. They are expressed in terms of *operations* that take one or several elements of the set, and return a result that may or may not be part of the given set. When an operation f takes an element of a set A and returns another element of A we write $f : A \to A$. When an operation $\circ$ takes two elements of a set A and returns a single element of A we write[1] $\circ : A \times A \to A$. Equipped with the mathematical tool of vectors we will explore the physical concepts of forces and torques, and how they are balanced in systems at rest.

[1] Here $A \times A$ is the set, (a_1, a_2), of all pairs of elements $a_1, a_2 \in A$. Further details will be given in Definition 2.3 below.

2.2 Sets

In mathematics and physics we often wish to make statements about a collection of objects, numbers, or other distinct entities.

Definition 2.1: Set

A *set* is a gathering of well-defined, distinct objects of our perception or thoughts.
An object a that is part of a set A is an *element* of A;
we write $a \in A$.
If a set M has a finite number n of elements we say that its *cardinality* is n. We write $|M| = n$.

Remark 2.1. Notations and additional properties:

a) When a set M has a finite number of elements, e.g., $+1$ and -1, one can specify the elements by explicitly stating the elements, $M = \{+1, -1\}$. In which order they are states does not play a role, and it also does not make a difference when elements are provided several times. In other words the set M of cardinality two can be specified by any of the following statements

$$M = \{-1, +1\} = \{+1, -1\} = \{-1, 1, 1, 1, \} = \{-1, 1, +1, -1\}$$

b) If e is not an element of a set M, we write $e \notin M$. For instance $-1 \in M$ and $2 \notin M$.

c) There is exactly one set with no elements, i. e. with cardinality zero. It is denoted as empty set, $\emptyset$.

Example 2.2: Sets

- Set of capitals of German states:

  $A_C = \{$Berlin, Bremen, Hamburg, Stuttgart, Mainz, Wiesbaden, München, Magdeburg, Saarbrücken, Potsdam, Kiel, Hannover, Dresden, Schwerin, Düsseldorf, Erfurt$\}$

- Set of small letters in German:

A_L = {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z, ä, ö, ü, ß}

- Set of month with 28 days:[2]

  A_M = {January, February, March, April, May, June, July, August, September, October, November, December}

The cardinalities of these sets are
$|A_C| = 16$, $|A_L = 30|$, and $|A_M| = 12$.

[2] Most of them have even more days.

Example 2.3: Sets of sets

A set can be an element of a set. For instance the set

$$M = \{1, 3, \{1, 2\}\}$$

has three elements 1, 3 and {1, 2} such that $|M| = 3$, and

$$1 \in M\,, \qquad \{1, 2\} \in M\,, \qquad 2 \notin M \qquad \{1\} \notin M\,.$$

Often it is bulky to list all elements of a set. In obvious cases we use ellipses such as A_L = {a, b, c, ..., z, ä, ö, ü, ß} for the set given in Example 2.2. Alternatively, one can provide a set M by specifying the properties $A(x)$ of its elements x in the following form

$$\underbrace{M}_{\text{The set } M} \underbrace{=}_{\text{contains}} \underbrace{\{}_{\text{all elements}} \underbrace{x}_{x,} \underbrace{:}_{\text{with :}} \underbrace{A(x)}_{\text{properties} \dots} \}.$$

where the properties specify one of several properties of the elements. The properties are separated by commas, and must all be true for all elements of the set.

Example 2.4: Set definition by property

The set of digits $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ can also be defined as follows $D = \{0, \dots, 9\} = \{x : 0 < x \leq 9, x \in \mathbb{Z}\}$. In the latter definition $\mathbb{Z}$ denotes the set of all integer numbers.

In order to specify the properties in a compact form we use logical junctors as short hand notation. In the present course we adopt the notations not $\neg$, and $\wedge$, or $\vee$, implies $\Rightarrow$, and is equivalent $\Leftrightarrow$ for the relations indicated in 2.1.

The definition of the digits in Example 2.4 entails that all elements of D are also numbers in $\mathbb{Z}$: we say that D is a subset of $\mathbb{Z}$.

Definition 2.2: Subset and Superset

The set M_1 is a *subset* of M_2, if all elements of M_1 are also contained in M_2. We write[3] $M_1 \subseteq M_2$. We denote M_2 then as *superset* of M_1, writing $M_2 \supseteq M_1$.
The set M_1 is a *proper subset* of M_2 when at least one of its

[3] Some authors use $\subset$ instead of $\subseteq$, and $\subsetneq$ to denote proper subsets.

Table 2.1: List of the results of different junctors acting on two statements A and B. Here 0 and 1 indicate that a statement is wrong or right, respectively. In the rightmost column we state the contents of the expression in the left column in words. The final three lines provide examples of more complicated expressions.

A	0	0	1	1	
B	0	1	0	1	
$\neg A$	1	1	0	0	not A
$\neg B$	1	0	1	0	not B
$A \vee B$	0	1	1	1	A or B
$A \wedge B$	0	0	0	1	A and B
$A \Rightarrow B$	1	1	0	1	A implies B
$A \Leftrightarrow B$	1	0	0	1	A is equivalent to B
$A \vee \neg B$	1	0	1	1	A or not B
$\neg A \wedge B$	0	1	0	0	not A or B
$A \wedge \neg B$	0	0	1	0	A and not B

elements is not contained in M_2. In this case $|M_1| < |M_2|$ and we write $M_1 \subset M_2$, or $M_2 \supset M_1$.

Example 2.5: Subsets

- The set of month with names that end with "ber" is a subset of the set A_M of Example 2.2

$$\{\text{September, October, November, December}\} \subseteq A_M$$

- For the set M of Example 2.3 one has

$$\{1\} \subseteq M\,, \quad \{1, 3\} \subseteq M\,, \quad \{1, 2\} \not\subseteq M\,, \quad \{2, \{1, 2\}\} \not\subseteq M\,.$$

Note that $\{1, 2\}$ is an elements of M. However, it is not a subset. The last two sets are no subsets because $2 \notin M$.

Two sets are the same when they are subsets of each other.

Theorem 2.1: Equivalence of Sets

Two sets A and B are *equal* or *equivalent*, iff

$$(A \subseteq B) \;\wedge\; (B \subseteq A)\,.$$

Remark 2.2 (iff). In mathematics "iff" indicates that something holds "if and only if". Observe its use in the following two statements: A number is an even number if it is the product of two even numbers. A number is an even number iff it is the product of an even number and another number.

Remark 2.3 (precedence of operations in logical expressions.). In logical expressions we first evaluate $\in$, $\notin$ and other set operations that are used to build logical expressions. Then we evaluate the junctor $\neg$ that is acting on a a single logical expression. Finally the other junctors $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are evaluated. Hence, the brackets are not required in Theorem 2.1.

Proof of Theorem 2.1.

$A \subseteq B$ implies that $a \in A \Rightarrow a \in B$.
$B \subseteq A$ implies $b \in B \Rightarrow b \in A$.
If $A \subseteq B$ and $B \subseteq A$, then we also have $a \in A \Leftrightarrow a \in B$. □

The description of sets by properties of its members, Example 2.4, suggests that one will often be interested in operations on sets. For instance the odd and even numbers are subsets of the natural numbers. Together they form this set, and one is left with the even numbers when removing the odd numbers from the natural numbers. Hence, we define the following operations on sets.

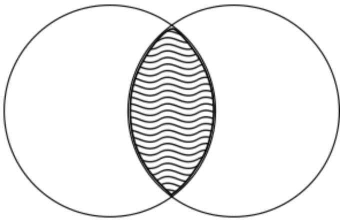


Figure 2.2: Intersection of two sets.

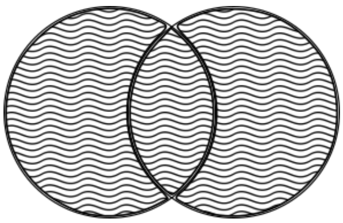


Figure 2.3: Union of two sets.

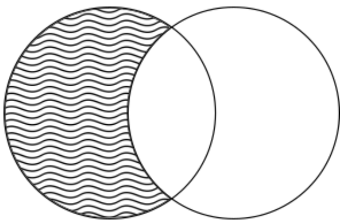


Figure 2.4: Difference of two sets.


Figure 2.5: Complement of a set.

Definition 2.3: Set Operations

For two sets M_1 and M_2 we define the following operations:

- *Intersection*: $M_1 \bigcap M_2 = \{m \,:\, m \in M_1 \wedge m \in M_2\}$,
- *Union*: $M_1 \bigcup M_2 = \{m \,:\, m \in M_1 \vee m \in M_2\}$,
- *Difference*: $M_1 \backslash M_2 = \{m \,:\, m \in M_1 \wedge m \notin M_2\}$,
- The *complement* of a set M in a *universe* U is defined for subsets $M \subseteq U$ as $M^C = \{m \in U \,:\, m \notin M\} = U \backslash M$.
- The *Cartesian product of two sets* M_1 *and* M_2 is defined as the set of ordered pairs (a, b) of elements $a \in M_1$ and $b \in M_2$, $M_1 \times M_2 = \{(a, b) \,:\, a \in M_1, b \in M_2\}$.

A graphical illustration of the operations is provided in Figures 2.2 to 2.5.

Example 2.6: Set operations: participants in my class

Consider the set of participants P in my class. The sets of female F and male M participants of the class are proper subsets of P with an empty intersection $F \bigcap M$. The set of non-female participants is $P \backslash F$. The set of heterosexual couples in the class is a subset of the Cartesian product $F \times M$. Furthermore, the union $F \bigcup M$ is a proper subset of P, when there is a participant who is neither female nor male.

2.2.1 *Sets of Numbers*

Many sets of numbers that are of interest in physics have infinitely many elements. We construct them in Table 2.2 based on the natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

or the natural numbers with zero

$$\mathbb{N}_0 = \mathbb{N} \bigcup \{0\} \,.$$

name	symbol	description
natural numbers	$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
natural numbers with 0	$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
negative numbers	$-\mathbb{N}$	$\{-n \,:\, n \in \mathbb{N}\}$
even numbers	$2\mathbb{N}$	$\{2n \,:\, n \in \mathbb{N}\}$
odd numbers	$2\,\mathbb{N} - 1$	$\{2n - 1 \,:\, n \in \mathbb{N}\}$
integer numbers	$\mathbb{Z}$	$(-\mathbb{N}) \cup \mathbb{N}_0$
rational numbers	$\mathbb{Q}$	$\left\{\frac{p}{q} \,:\, p \in \mathbb{Z}, q \in \mathbb{N}\right\}$
real numbers	$\mathbb{R}$	see below
complex numbers	$\mathbb{C}$	$\mathbb{R} + \mathrm{i}\mathbb{R}$, where $\mathrm{i} = \sqrt{-1}$

Table 2.2: Summary of important sets of numbers.

Remark 2.4. Some authors adopt the convention that zero is included in the natural numbers $\mathbb{N}$. When this matters you have to check which convention is adopted.

There are many more sets of numbers. For instance, in mathematics the set of constructable numbers is relevant for certain proofs in geometry, and in physics and computer graphics quaternions are handy when it comes to problems involving three-dimensional rotations. In any case one needs intervals of numbers.

Definition 2.4: Interval of Real Numbers $\mathbb{R}$

An *interval* is a continuous subset of a set of numbers. We distinguish *open, closed, and half-open subsets*.

- closed interval: $[a, b] = \{x \,:\, x \geq a, x \leq b\}$,
- open interval: $(a, b) =]a, b[= \{x \,:\, x > a, x < b\}$,
- right open interval: $[a, b) = [a, b[= \{x \,:\, x \geq a, x > b\}$,
- left open interval: $(a, b[=]a, b] = \{x \,:\, x > a, x \geq b\}$.

Subsets of $\mathbb{R}$ will be denoted as real intervals.

add limits, closure, and $\mathbb{R}$ as closure of $\mathbb{Q}$.

2.2.2 Self Test

Problem 2.1. **Relations between sets**

Let A, B, C, and D be pairwise distinct elements. Select one of the symbols

$$\in, \quad \notin, \quad \ni, \quad \not\ni, \quad \subset, \quad \not\subset, \quad \supset, \quad \not\supset, \quad =$$

and avoid $\notin$, $\not\ni$, $\not\subset$, $\not\supset$ wherever possible.

a) $\{A, B\} \;\square\; \{A, B, C\}$,

b) $\{A\} \;\square\; B$,

c) $\{\varnothing\} \;\square\; \varnothing$,

d) $\{\{A\}\} \;\square\; \{\{A\}, \{B\}\}$,

e) $A \;\square\; \{A, B, C\}$,

f) $\{A, C, D\} \cap \{A, B\} \;\square\; \{A, B, C, D\}$,

g) $\{A, C, D\} \setminus \{A, B\} \;\square\; \{A, B, C\}$,

h) $\{A, C, D\} \cup \{A, B\} \;\square\; A$.

Problem 2.2. **Intervals**

a) Provide $[1;17]\cap]0;5[$ as a single interval.

b) Provide $[-1,4]\setminus[1,2[$ as union of two intervals.

Problem 2.3. **Sets of numbers**

Which of the following statements are true?

a) $\{6 \cdot z | z \in \mathbb{Z}\} \subset \{2 \cdot z | z \in \mathbb{Z}\}$.

b) $\{2 \cdot z | z \in \mathbb{Z}\} \cap \{3 \cdot z | z \in \mathbb{Z}\} = \{6 \cdot z | z \in \mathbb{Z}\}$.

c) Let $T(a)$ be the set of numbers that divide a. Then

$$\forall a, b \in \mathbb{N} : \quad T(a) \cup T(b) = T(a \cdot b)$$

Example: $T(2) = \{1,2\}$, $T(3) = \{1,3\}$, and $T(6) = \{1,2,3,6\}$.

2.3 *Groups*

A group G refers to a set of operations $t \in G$ that are changing some data or objects. Elementary examples refer to reflections in space, turning some sides of a Rubik's cube, or translations in space, as illustrated in Figure 2.1. The subsequent action of two group elements t_1 and t_2 of G is another (typically more complicated) transformation $t_3 \in G$. Analogous to the concatenation of functions, we write $t_3 = t_2 \circ t_1$, and we say t_3 is t_2 after t_1. The set of transformations forms a group iff it obeys the following rules.

Definition 2.5: Group

A set $(G, \circ)$ is called a *group* with operation $\circ : G \times G \to G$ when the following rules apply

a) The set is *closed*: $\forall g_1, g_2 \in G : g_1 \circ g_2 \in G$.

b) The set has a *neutral element*: $\exists e \in G\ \forall g \in G : e \circ g = g$.

c) Each element has an *inverse element*:
$$\forall g \in G\ \exists i \in G : g \circ i = e.$$

d) The operation $\circ$ is *associative*:
$$\forall g_1, g_2, g_3 \in G : (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3).$$

Definition 2.6: Commutative Group

A group $(G, \circ)$ is called a *commutative group* when

e) the group operation is *commutative*:
$$\forall g_1, g_2 \in G : g_1 \circ g_2 = g_2 \circ g_1.$$

When the group has a finite number of elements the result of the group operation can explicitly be specified by a group table. We demonstrate this by the smallest groups. The empty set can not be a group because it has no neutral element. Therefore the smallest groups have a single element and two elements. Both of these groups are commutative.

Example 2.7: Smallest groups

$(\{n\}, \odot)$ comprises only the neutral element.

$$\begin{array}{c|c} \odot & n \\ \hline n & n \end{array}$$

The smallest non-trivial group has two elements $(\{0,1\}, \oplus)$ with

$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

It describes the turning of a piece of paper:
Not turning, 0, does not change anything (neutral element).
Turning, 1, shows the other side, and turning twice is equivalent to not turning at all (1 is its own inverse).

Remark 2.5. The group properties imply that all elements of the group must appear exactly once in each row and each column of the group table. As a consequence the smallest non-commutative group is the dihedral group of order 6 with six elements that is discussed in Problem 2.6.

Example 2.8: Non-commutative groups: rotations

The rotation of an object in space is a group. In particular this holds for the 90°-rotations of an object around a vertical and a horizontal axis. Figure 2.6 illustrates that these rotations do not commute.

Figure 2.6: Rotation of a book by multiples of $\pi/2$ around three orthogonal axes.

Example 2.9: Non-commutative groups: edit text fields

We consider the text fields of a fixed length n in an electronic form. Then the operations
"Put the letter L into position $\sqcup$ of the field"
with $L \in \{\_, a, \ldots, z, A, \ldots, Z\}$
and $\sqcup \in \{1, \ldots, n\}$ form a group.
Also in this case one can easily check that the order of the operations is relevant. In the left and right column the same

operations are preformed for a text field of length $n = 4$:

$$\begin{array}{rr}
|_|_|_|_| & |_|_|_|_| \\
\to |M|_|_|_| & \to |P|_|_|_| \\
\to |M|a|_|_| & \to |P|h|_|_| \\
\to |M|a|t|_| & \to |P|h|y|_| \\
\to |M|a|t|h| & \to |P|h|y|s| \\
\to |M|a|t|s| & \to |P|h|y|h| \\
\to |M|a|y|s| & \to |P|h|t|h| \\
\to |M|h|y|s| & \to |P|a|t|h| \\
\to |P|h|y|s| & \to |M|a|t|h|
\end{array}$$

Remark 2.6. Notations and additional properties:

a) Depending of the context the inverse element is denoted as g^{-1} or as $-g$. This depends on whether the operation is considered a multiplication or rather an addition. In accordance with this choice the neutral element is denoted as 1 or 0.

b) The second property of groups, b) $\exists e \in G\ \forall g \in G : e \circ g = g$, implies that also $g \circ e = g$. The proof is provided as Problem 2.5.

example?

c) When a group is not commutative then one must distinguish the left and right inverse. The condition $g \circ i = e$ does not imply $i \circ g = e$. However, there always is another element $j \in G$ such that $j \circ g = e$.

2.3.1 *Self Test*

Problem 2.4. Checking group axioms
Which of the following sets are groups?

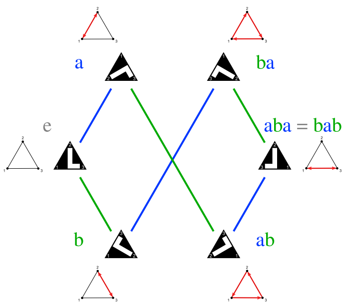
a) $(\mathbb{N}, +)$ c) $(\mathbb{Z}, \cdot)$ e) $(\{0\}, +)$

b) $(\mathbb{Z}, +)$ d) $(\{+1, -1\}, \cdot)$ $(\{1, \ldots, 12\}, \oplus)$

where $\oplus$ in f) revers to adding as we do it on a clock, e.g. $10 \oplus 4 = 2$.

Problem 2.5. **Uniqueness of the neutral element**
Proof that the group axioms, Definition 2.5, imply that $e \circ g = g$ implies that also $g \circ e = g$.

Problem 2.6. **Dihedral group of order** 6
Figure 2.7 illustrates the effect of reflections of a triangle with respect to its three symmetry axis. All group elements can be generated by repeated action of two reflections, e.g. those denoted as a and b in the figure.



Watchduck (a.k.a. Tilman Piesk), wikimedia CC BY-SA
Figure 2.7: Reflections of equilateral triangle with respect to the three symmetry axes form a group with six elements; see Problem 2.6.

a) Verify that the group properties, Definition 2.5, together with the three additional requirements

$$a \circ a = b \circ b = e \text{ and } a \circ b \circ a = b \circ a \circ b$$

imply that the group has exactly three elements, $\{e, a, b, a \circ b, b \circ a, a \circ b \circ a\}$.

b) Work out the group table.

2.4 Fields

Besides being of importance to characterize the action of symmetry operations like reflections or rotations, groups are also important for us because they admit further characterization of sets of numbers.

The natural numbers are not a group. For the addition they are lacking the neutral elements, and for adding and multiplications they are lacking inverse elements.

In contrast the group $(\mathbb{Z}, +)$ is a commutative group with infinitely many elements.

Example 2.10: The group $(\mathbb{Z}, +)$

The numbers $\mathbb{Z}$ with operation $+$ form a group. This is demonstrated here by checking the group axioms.

a) Addition of any two numbers provides a number:
$\forall x, y \in \mathbb{Z} \, : \, (x + y) \in \mathbb{Z}$.

b) The neutral element of the addition is 0:
$\exists\, 0 \in \mathbb{Z} \; \forall z \in \mathbb{Z} \, : \, z + 0 = z = 0 + z$.

c) For every element $z \in \mathbb{Z}$ there is an inverse $(-z) \in \mathbb{Z}$:
$\forall z \in \mathbb{Z} \; \exists (-z) \in \mathbb{Z} \, : \, z + (-z) = 0 = (-z) + z$.

d) The addition of numbers is associative:
$\forall z_1, z_2, z_3 \in \mathbb{Z} \, : \, z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

However, the numbers $\mathbb{Z}$ still lack inverse elements of the multiplication. The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are a commutative groups for addition and multiplication (with the special rule that multiplication with 0 has no inverse element), and their elements also obey distributivity. Such sets are called number fields.

Definition 2.7: Field

A set $(\mathbb{F}, +, \cdot)$ is called a *field* with neutral elements 0 and 1 for addition $+$ and multiplication $\cdot$, respectively, when its elements comply with the following rules

a) $(\mathbb{F}, +)$ is a commutative group,

b) $(\mathbb{F}\backslash\{0\}, \cdot)$ is a commutative group,

c) Addition and Multiplication are distributive:

$$\forall a, b, c \in \mathbb{F} : a \cdot (b + c) = a \cdot b + a \cdot c$$

Remark 2.7. For the multiplication of field elements one commonly suppresses the $\cdot$ for the multiplication, writing e.g. $a\,b$ rather than $a \cdot b$.

Example 2.11: The smallest field has two elements

The smallest field $(\{0, 1\}, \oplus, \odot)$ comprises only the neutral elements 0 of the group $(\{0, 1\}, \oplus)$ with two elements, and 1 of the group $(\{1\}, \odot)$ with one element.

Example 2.12: Complex numbers are a field

a) The sum of two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ amounts to

$$z_1 + z_2 = (x_1 + \mathrm{i}y_1) + (z_2 = x_2 + \mathrm{i}y_2) = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

Hence, the group properties for $+$ follow from the properties of the real numbers x_1, x_2 and y_1, y_2, respectively.
c) They also entail distributivity of complex numbers.
b) The product of the complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ amounts to

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + \mathrm{i}y_1) \cdot (z_2 = x_2 + \mathrm{i}y_2) \\ &= (x_1\,x_2 + \mathrm{i}y_1\,x_2 + \mathrm{i}y_1\,x_2 + \mathrm{i}^2\,y_1\,y_2) \\ &= (x_1\,x_2 - y_1\,y_2) + \mathrm{i}\,(y_1\,x_2 + x_1\,y_2) \end{aligned}$$

Checking the group axioms based on this representation of the complex numbers is tedious. One better adopts a representation in terms of polar coordinates, $z_1 = R_1\,\mathrm{e}^{\mathrm{i}\varphi_1}$ and $z_2 = R_2\,\mathrm{e}^{\mathrm{i}\varphi_2}$ (see Figure 2.8) where (cf Problem 2.8)

$$z_1 \cdot z_2 = R_1\,\mathrm{e}^{\mathrm{i}\varphi_1} \cdot R_2\,\mathrm{e}^{\mathrm{i}\varphi_2} = \big(R_1\,R_2\big)\,\mathrm{e}^{\mathrm{i}\,(\varphi_1+\varphi_2)}$$

Here, the group properties follow from those of multiplying R_1 and R_2, and adding φ_1 and φ_2.

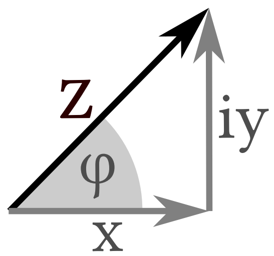


Figure 2.8: Complex numbers z can be represented as $z = x + \mathrm{i}y$ in a plane where (x, y) are the Cartesian coordinates of z. Alternatively, one can adopt a representation in terms of polar coordinates $z = R\,\mathrm{e}^{\mathrm{i}\varphi}$ where $R = \sqrt{x^2 + y^2}$ and φ is the angle with respect to the x-axis.

Remark 2.8 (complex conjugation). Each complex numbers z has a complex conjugate, denoted as z^* or $\bar{z}$, that is defined as

$$\forall z = x + \mathrm{i}y = R\,\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C} : \quad \bar{z} = x - \mathrm{i}y = R\,\mathrm{e}^{-\mathrm{i}\varphi} \tag{2.4.1}$$

Complex conjugation provides an effective way to calculate the absolute value $|z| = R$ of complex numbers

$$\begin{aligned}
& z\,\bar{z} = (x + \mathrm{i}y)\,(x - \mathrm{i}y) = x^2 - \mathrm{i}^2\,y^2 = x^2 + y^2 = R^2 \\
\text{and} \quad & z\,\bar{z} = R\,\mathrm{e}^{\mathrm{i}\varphi}\,R\,\mathrm{e}^{-\mathrm{i}\varphi} = R^2\,\mathrm{e}^0 = R^2 \\
\Rightarrow \quad & |z| = \sqrt{z\,\bar{z}} = \sqrt{\bar{z}\,z}
\end{aligned} \tag{2.4.2}$$

Remark 2.9. In physics complex numbers are commonly applied to describe rotations in a plain: Multiplication by $\mathrm{e}^{\mathrm{i}\theta}$ rotates a complex number z by an angle θ around the origin:

$$\forall z = R\,\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}: \quad z \cdot \mathrm{e}^{\mathrm{i}\theta} = R\,\mathrm{e}^{\mathrm{i}(\varphi+\theta)} \tag{2.4.3}$$

2.4.1 *Self Test*

Problem 2.7. Checking field axioms
Which of the following sets are fields?

a) $(\mathbb{Z}, +, \cdot)$

b) $(\{1, 2, \ldots, 12\}, +\mathrm{mod}12, \cdot\mathrm{mod}12)$
like on a clock: $11 + 2 = 13\mathrm{mod}12 = 1$ and $4 \cdot 5 = 20\mathrm{mod}12 = 8$.

c) $(\{0, 1, 2\}, +\mathrm{mod}3, \cdot\mathrm{mod}3)$
for instance $2 \cdot 2 = 2 + 2 = 4\mathrm{mod}3 = 1$ and $2 + 1 = 3\mathrm{mod}3 = 0$.

Problem 2.8. **Euler's equation and trigonometric relations**
Euler's equation $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\,\sin x$ relates complex values exponential functions and trigonometric functions.

a) Sketch the position of $R\,\mathrm{e}^{\mathrm{i}x}$ in the complex plain, and indicate how Euler's equation is related to the Theorem of Pythagoras.

b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular, for $R = 1$ one has $\mathrm{e}^{\mathrm{i}\,(x+y)} = \mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{\mathrm{i}y}$. Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about $\sin(2x)$ and $\cos(2x)$?

2.5 *Vector Spaces*

With the notions introduced in the preceding sections we can give now the formal definition of a vector space

Definition 2.8: Vector Space

A *vector space* $(\mathsf{V}, \mathbb{F}, \oplus, \odot)$ is a set of *vectors* $\boldsymbol{v} \in \mathsf{V}$ over a field $(\mathbb{F}, +, \cdot)$ with binary operations $\oplus : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ and $\odot : \mathbb{F} \times \mathsf{V} \to \mathsf{V}$ complying with the following rules

a) $(\mathsf{V}, \oplus)$ is a commutative group

b) associativity: $\forall a, b \in \mathbb{F}\ \forall \boldsymbol{v} \in \mathsf{V} : a \odot (b \odot \boldsymbol{v}) = (a \cdot b) \odot \boldsymbol{v}$

c) distributivity 1:

$$\forall a, b \in \mathbb{F}\ \forall \boldsymbol{v} \in \mathsf{V} : (a+b) \odot \boldsymbol{v} = (a \odot \boldsymbol{v}) \oplus (b \odot \boldsymbol{v})$$

d) distributivity 2:

$$\forall a \in \mathbb{F}\ \forall \boldsymbol{v}, \boldsymbol{w} \in \mathsf{V} : a \odot (\boldsymbol{v} \oplus \boldsymbol{w}) = (a \odot \boldsymbol{v}) \oplus (a \odot \boldsymbol{w})$$

Remark 2.10. It is common to use $+$ and $\cdot$ instead of $\oplus$ and $\odot$, respectively, with the understanding that it is clear from the context in the equation whether the symbols refer to operations involving vectors, only numbers, or a number and a vector.

Moreover, as for the multiplication of numbers, one commonly drops the $\odot$ for the multiplication, writing e.g. $a\,\boldsymbol{v}$ rather than $a \cdot \boldsymbol{v}$.

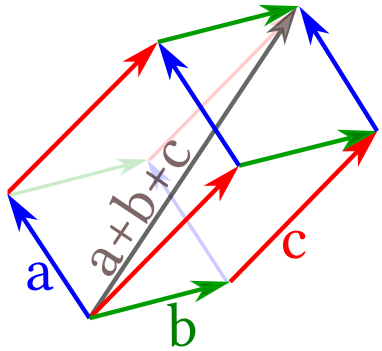


Figure 2.9: The arrows indicate displacements by three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, as discussed in Example 2.13. Their commutativity and associativity follow from the properties of parallelograms. This holds in the plane, and also when the vectors span a three-dimensional volume.

Example 2.13: Vector spaces: displacements in the plane

For displacements we define the operation $\oplus$ as concatenation of displacements, and $\odot$ as increasing the length of the displacement by a given factor without touching the direction.

a) The neutral element amounts to staying, one can always shift back, move between any two points in a plane, and commutativity follows form the properties of parallelograms, see Figure 2.9.

b,c) The vectors selects the direction, and scalar multiplications only work on its length, i.e. a real number.

d) Is implied by the Intersept Theorem.

Example 2.14: Vector spaces: $\mathbb{R}^D$

For every $D \in \mathbb{N}$ the D-fold Cartesian product $\mathbb{R}^D$ of the real numbers is a vector space over $\mathbb{R}$ when defining the operation $+$ and $\cdot$ as

$$\forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^D : \boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_D \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_D + b_D \end{pmatrix}$$

$$\forall s \in \mathbb{R}\ \forall \boldsymbol{a} \in \mathbb{R}^D : s \cdot \boldsymbol{a} = s \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{pmatrix} = \begin{pmatrix} s\,a_1 \\ s\,a_2 \\ \vdots \\ s\,a_D \end{pmatrix}$$

In a more compact manner this is also written as,

$$\forall \boldsymbol{a} = (a_i), \boldsymbol{b} \in (b_i), s \in \mathbb{R} : \boldsymbol{a} + \boldsymbol{b} = (a_i + b_i) \ \wedge\ s\,\boldsymbol{a} = (s\,a_i)$$

Checking the properties of a vector space is given as Problem 2.9a).

Definition 2.9: $N \times M$ Matrix: $\mathbb{M}^{M\times N}(\mathbb{F})$

For $N, M \in \mathbb{N}$ we define $N \times M$ *matrices* A, B, ... over the field $\mathbb{F}$ as arrays, $A = (a_{ij})$, $B = (b_{ij})$, with components $a_{ij}, b_{ij}, \in \mathbb{F}$. The indices $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$ label the rows and columns of the array, respectively.
The sum of matrices and the product with a scalar are defined component-wise as

$$\forall A, B \in \mathbb{M}^D,\ c \in \mathbb{F} : A + B = \left(a_{ij} + b_{ij}\right) \ \wedge\ c \cdot A = \left(c\, a_{ij}\right)$$

Example 2.15: Vector spaces: $M \times N$ matrices

The $M \times N$ matrices over a field $\mathbb{F}$, $(\mathbb{M}^{M\times N}, \mathbb{F}, +, \cdot)$ form a vector space. The proof is given as Problem 2.9b).

Definition 2.10: Vector space: $N \times M$ matrices

For matrices one defines a product as follows

$$\odot : \mathbb{M}^{M\times L} \times \mathbb{M}^{L\times N} \to \mathbb{M}^{M\times N}$$

$$\forall A \in \mathbb{M}^{M\times L}, B \in \times\mathbb{M}^{L\times N} : A \odot B = C = (c_{ij}) = \left(\sum_{k=1}^{L} a_{ik} b_{kj}\right)$$

Remark 2.11. Also for matrix multiplication one commonly suppresses the $\oplus$ operator, writing $A\,B$ rather than $A \odot B$.

Remark 2.12. For square matrices $\mathbb{M}^{M\times M}$ the operation $+$ and $\odot$ define a sum and a product that take two elements of $\mathbb{M}^{M\times M}$ and return an element of $\mathbb{M}^{M\times M}$. Nevertheless, $(\mathbb{M}^{M\times M}, +, \odot)$ is *not* a field: In general, $\odot$ is not commutative and matrices do not necessarily have an inverse.

Example 2.16: Vector spaces: Polynomials of degree 2

For a field $\mathbb{F}$ the polynomials P_2 of degree two in the variable x are defined as

$$\mathsf{P}_2 = \{\boldsymbol{p} = [p_0 + p_1\, x + p_2\, x^2] : p_0, p_1, p_2 \in \mathbb{F}\}$$

This set is a vector space with respect to the summation

$$\begin{aligned}\boldsymbol{p} + \boldsymbol{q} &= [p_0 + p_1\, x + p_2\, x^2] + [q_0 + q_1\, x + q_2\, x^2] \\ &= \Big[(p_0 + q_0) + (p_1 + q_1)\, x + (p_2 + q_2)\, x^2\Big]\end{aligned}$$

and the multiplication with a scalar $s \in \mathbb{F}$

$$s \cdot \boldsymbol{p} = s \cdot (p_0 + p_1\, x + p_2\, x^2) = \Big[(s\, p_0) + (s\, p_1)\, x + (s\, p_2)\, x^2)\Big]$$

Proof. Each element $\boldsymbol{p} = [p_0 + p_1\, x + p_2\, x^2]$ of this vector space is uniquely described by the three-tuple $(p_0, p_1, p_2) \in \mathbb{F}^3$ with rules for addition and scalar multiplication analogous to those discussed for $\mathbb{R}^3$ in Example 2.14. Hence, the proof for $\mathbb{R}^3$ also applies here. □

In physics we heavily make use of the correspondence evoked by the proof in Example 2.16. The relative position of two objects with respect to each other is commonly described in terms of (the sum of several) vectors. In order to gain further information about the positions, we will then recast the *geometric* problem about the positions into an *algebraic* problem stated in terms of linear equations. The latter can then be solved by straightforward analytical calculations. Vice versa, abstract findings about the solutions of sets of equations will be recast in terms of geometry in order to visualize the abstract results. The change of perspective has become a major avenue to drive theoretical physics throughout the 20th century. For mechanical problems it forms the core of the mathematical formulation of problems in robotics and computer vision. Quantum mechanics is entirely build on the principles of vector spaces and their generalization to Hilbert spaces. General relativity and quantum field theory take Noether's theorem as their common starting point, which is build upon concepts from group theory and the requirement that physical predictions must not change when taking different choices how to mathematically describe the system. An important concern of these notes is to serve as a training ground to practice the changing of mathematical perspective for the purpose of solving physics problem. As a first physical application we discuss now force balances. Then we resume the discussion of vector spaces, taking a closer look into the calculation of coordinates and distances.

2.5.1 *Self Test*

Problem 2.9. **Checking vector-space properties**

a) Verify that $\mathbb{R}^D$ with the operations defined in Example 2.14 is a vector space.

b) Verify that $N \times M$ matrices, as defined in Definition 2.9, form a vector space.

Problem 2.10. **Polynomials of degree** N

For a field $\mathbb{F}$ the polynomials P_N of degree N in the variable x are defined as

$$\mathsf{P}_N = \left\{ \boldsymbol{p} = \left[\sum_{i=0}^{N} p_i \, x^i \right] : p_0, \ldots, p_N \in \mathbb{F} \right\}$$

a) State the rules of addition and multiplication with a scalar $s \in \mathbb{F}$ in analogy to the special case of $N = 2$ discussed in Example 2.16.

b) Verify that the polynomials of degree N are a vector space.

2.6 Balancing Forces

It is an experience from tug of war that nothing moves as long as forces are balanced. In this example one can add a ring to the rope. The pulling forces act in opposing directions on the ring, as illustrated in the upper left diagram in Figure 2.10. The lower left diagram shows the case, where three parties are pulling on the ring. In any case the total force on the ring amounts to the sum of the acting forces, forces are vectors, and all sums of vectors obeys the same rules. As far as graphical illustrations are concerned the sum of forces looks therefore the same as the sum of displacements in Figure 2.1. For the ring the sums of the forces are illustrated in the right panels of Figure 2.10. The ring does not move when they add to zero.

Tug of War, Nikolay Bogdanov-Belsky, 1939 wikiart / public domain

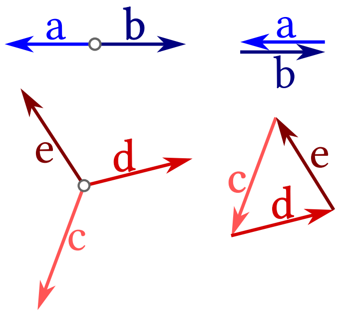


Figure 2.10: The left diagrams show two and three forces acting on a ring. To the right it is demonstrated that they add to zero.

Axiom 2.1: Force balance

Let N forces $\boldsymbol{F}_1, \dots, \boldsymbol{F}_N$ act on a body. The body does not move as long as the forces add to zero, i.e. iff $\mathbf{0} = \sum_{i=1}^{N} \boldsymbol{F}_i$.

Remark 2.13. Strictly speaking the body might turn, but its center of mass will not move. We come back to this point in Section 2.9.

explain center of mass

Example 2.17: Balancing on a slackline

A person balances on a slackline that is fixed to trees at its opposing sides. At the point where she is standing there are three forces acting:
her weight $\boldsymbol{F}_d = M\boldsymbol{g}$ pushing downwards, and
forces along the slackline towards the left $\boldsymbol{F}_l$ and right $\boldsymbol{F}_r$.
She can stay at rest as long as

$$\mathbf{0} = \boldsymbol{F}_d + \boldsymbol{F}_l + \boldsymbol{F}_r$$

The forces $\boldsymbol{F}_l$ and $\boldsymbol{F}_r$ are counterbalanced by the trees. These forces become huge when the slackline runs almost horizontally. Every now a then a careless slackliner roots out a tree or fells a pillar.

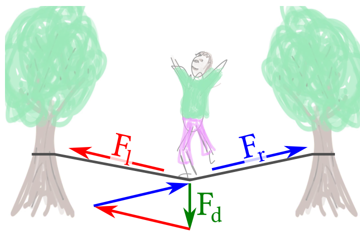


Figure 2.11: For a person balancing on a slackline, the graviational force $\boldsymbol{F}_d$ (d for down) is balanced by forces $\boldsymbol{F}_l$ and $\boldsymbol{F}_r$ along the line that pull towards the left and right, respectively. See Example 2.17 for futher discussion.

Example 2.18: Measuring the static friction coefficient

In a rough approximation static friction between two surfaces arises due to interlocking or surface irregularities. One must lift a block by a little amount to unlock the surfaces. In line with this argument dimensional analysis suggest that static friction should be proportional to the normal force between the surfaces. It is independent of the contact area, and depends on the material of the surfaces. This is indeed what is observed experimentally: The static friction force, $\boldsymbol{f}$ in Figure 2.12, can take values up to a maximum value of γ

times the normal force, $\boldsymbol{F}_N$, where γ typically takes values slightly less than one. By splitting the gravitational force, $m\boldsymbol{g}$ acting on a block on a plane into its components parallel and normal to the surface (gray arrows in Figure 2.12), one finds that in the presence of a force balance $m\boldsymbol{g} + \boldsymbol{f} + \boldsymbol{F}_N = \boldsymbol{0}$ one has

$$\left.\begin{aligned} F_N &= mg\,\cos\theta \\ f &= mg\,\sin\theta \\ f &< \gamma F_N \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow \sin\theta < \gamma\cos\theta \\ &\qquad \Rightarrow \theta < \theta_c = \arctan\gamma \end{aligned}$$

When θ exceeds θ_c the block starts to slide. Hence, one can infer γ from measurements of θ_c.

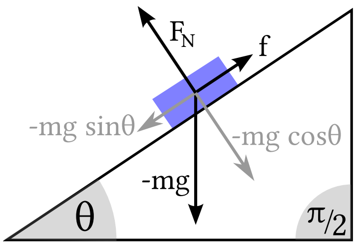
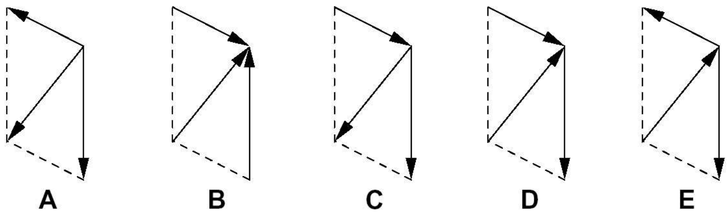


Figure 2.12: (top) As long as θ is smaller than the angle of friction the blue block does not slide. (bottom) Placing my cell phone on two rubber bands on a folder provides a maximum angle of about 33°, i. e. $\mu \simeq 0.5$. Using PhyPhox and a cell phone one can easily measure θ_c and μ for other combinations of materials.

2.6.1 *Self Test*

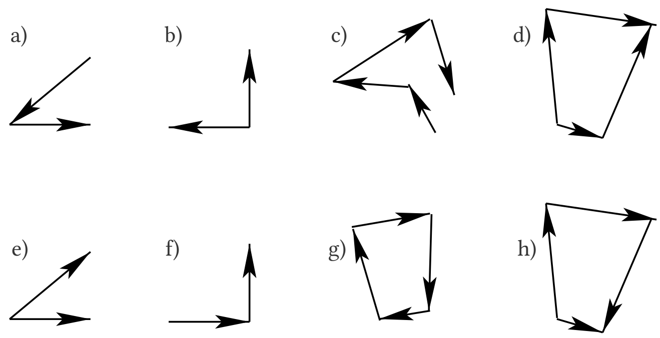
Problem 2.11. **Particles at rest**

There are three forces acting on the center of mass of a body. In which cases does it stay at rest?



Problem 2.12. **Graphical sum of vectors**

Determine the sum of the vectors. In which cases is the resulting vector vertical to the horizontal direction?



Problem 2.13. **Towing a Stone**

Three Scottish muscleman[4] try to tow a stone with mass $M = 20\,\text{cwt}$ from a field. Each of them gets his own rope, and he can act a maximal force of 300 lbf as long as the ropes run in directions that differ by at least 30°.

a) Sketch the forces acting on the stone and their sum. By which ratio is the force exerted by three men larger than that of a single man?

[4] In highland games one still uses Imperial Units. A hundredweight (cwt) amounts to eight stones (stone) that each have a mass of 14 pounds (lb). A pound-force (lbg) amounts to the gravitational force acting on a pound. One can solve this problem without converting units.

b) The stone counteracts the pulling of the men by a static friction force $\mu M g$, where g is the gravitational acceleration. What is the maximum value that the friction coefficient μ may take when the men can move the stone?

2.7 *The Inner Product*

The position of a particle, the direction of its motion and the angle of attack of forces are constantly changing during the motion of a particle. In Chapter 3 we explore how they are related. The calculations are feasible because the involved vector spaces also have an inner product.

Definition 2.11: Inner Product of a Vector Space

The *inner product* on a vector space $(\mathsf{V}, \mathbb{C}, \oplus, \odot)$ defines a binary operation $\langle \_ \mid \_ \rangle : \mathsf{V} \times \mathsf{V} \to \mathbb{F}$ with the following properties for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathsf{V}$ and $c \in \mathbb{C}$

a) conjugate symmetry: $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = \overline{\langle \boldsymbol{w} \mid \boldsymbol{v} \rangle}$

b) linearity in the first argument: $\langle c\,\boldsymbol{v} \mid \boldsymbol{w} \rangle = c\,\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$

   and $\langle \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{w} \rangle = \langle \boldsymbol{u} \mid \boldsymbol{w} \rangle + \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$

c) positivity: $\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle \geq 0$

   where equality applies iff $\boldsymbol{v} = \boldsymbol{0}$, $\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$

Remark 2.14. The idea underlying these properties is that $\sqrt{\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle}$ can be interpreted as the length of the vector $\boldsymbol{v}$.

Remark 2.15. Conjugate linearity and symmetry for the first argument imply the following relations for the second argument

$$\langle \boldsymbol{v} \mid c\,\boldsymbol{w} \rangle = \overline{\langle c\,\boldsymbol{w} \mid \boldsymbol{v} \rangle} = \bar{c}\,\overline{\langle \boldsymbol{w} \mid \boldsymbol{v} \rangle} = \bar{c}\,\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$$
$$\langle \boldsymbol{u} \mid \boldsymbol{v} + \boldsymbol{w} \rangle = \overline{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{u} \rangle} = \overline{\langle \boldsymbol{w} \mid \boldsymbol{u} \rangle} + \overline{\langle \boldsymbol{w} \mid \boldsymbol{v} \rangle} = \langle \boldsymbol{u} \mid \boldsymbol{w} \rangle + \langle \boldsymbol{u} \mid \boldsymbol{w} \rangle$$

Remark 2.16. Certain properties that hold for addition and scalar multiplication do *not* hold for the inner product.

a) There is no inverse: The information about the direction of vectors is lost upon taking the inner product. For instance, when $\langle \boldsymbol{u} \mid \boldsymbol{v} \rangle = 0$ and $\langle \boldsymbol{u} \mid \boldsymbol{w} \rangle = 0$ then one still can not tell the result of $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$.

b) Associativity does not hold: $\langle \boldsymbol{u} \mid \boldsymbol{v} \rangle\, \boldsymbol{w} \neq \boldsymbol{u}\, \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$.

Example 2.19: Inner product for real-valued vectors

For real-valued vectors the inner product is commutative, $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = \langle \boldsymbol{w} \mid \boldsymbol{v} \rangle$. The inner product is then also be written as $\boldsymbol{v} \cdot \boldsymbol{w}$, and it obeys bilinearity

$$\boldsymbol{u} \cdot (a\boldsymbol{v} + b\boldsymbol{w}) = a(\boldsymbol{u} \cdot \boldsymbol{v}) + b(\boldsymbol{u} \cdot \boldsymbol{w})$$

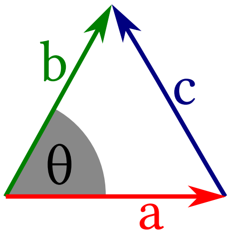


Figure 2.13: Notations for the geometric interpretation of the scalar product, Theorem 2.2

Theorem 2.2: Geometric Interpretation of the Inner Product for Real-Valued Vectors

For vectors of $\mathbb{R}^D$ the inner product of two vectors $\boldsymbol{a}, \boldsymbol{b}$ takes the value

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \, |\boldsymbol{b}| \, \cos\theta$$

where $\theta = \angle(\boldsymbol{a}, \boldsymbol{b})$ is the angle between the two vectors, see Figure 2.13.

Proof. The cosine theorem for triangles with sides of length a, b and c and angle θ opposite to c states that

$$c^2 = a^2 + b^2 - 2\,a\,b\,\cos\theta$$

Let now a, b, and c be the length of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$, as shown in Figure 2.13. Then we have

$$\begin{aligned} a^2 + b^2 - 2\,a\,b\,\cos\theta &= c^2 = \boldsymbol{c} \cdot \boldsymbol{c} = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) \\ &= \boldsymbol{a} \cdot \boldsymbol{a} - 2\,\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b} = a^2 + b^2 - 2\,\boldsymbol{a} \cdot \boldsymbol{b} \\ \Rightarrow \quad \boldsymbol{a} \cdot \boldsymbol{b} &= |\boldsymbol{a}| \, |\boldsymbol{b}| \, \cos\theta \end{aligned}$$

□

Remark 2.17. Theorem 2.2 entails that the scalar product $\boldsymbol{u} \cdot \boldsymbol{v}$ vanishes when the vectors are orthogonal, $\theta = \pi/2$. Also in general we say that

$$\boldsymbol{v} \text{ and } \boldsymbol{w} \text{ are othogonal iff} \qquad \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = 0\,.$$

Remark 2.18. The expression for the inner product that is provided Theorem 2.2 does not imply that the inner product is unique. Rather it is a consequence of the cosine theorem that holds iff the geometric interpretation of the vectors applies. This is demonstrated by an example provided in Problem 2.14.

2.7.1 *Self Test*

Problem 2.14. **The inner product is not unique**

Let v_1 and v_2 be two non-orthogonal vectors in a two-dimensional vector space with an inner product $\langle \_ \mid \_ \rangle$, and let λ_1 and λ_2 two positive real numbers. Then the following relation defines another inner product $(\_ \mid \_)$:

$$(\boldsymbol{a} \mid \boldsymbol{b}) = \lambda_1 \langle \boldsymbol{a} \mid \boldsymbol{e}_1 \rangle \, \langle \boldsymbol{e}_1 \mid \boldsymbol{b} \rangle + \lambda_2 \langle \boldsymbol{a} \mid \boldsymbol{e}_2 \rangle \, \langle \boldsymbol{e}_2 \mid \boldsymbol{b} \rangle \tag{2.7.1}$$

a) Verify that the properties a) and b) of an inner product $\langle \_ \mid \_ \rangle$ as given in Definition 2.11 are also be obeyed by $(\_ \mid \_)$.

b) Verify that $(\boldsymbol{a} \mid \boldsymbol{a}) \geq 0$ iff λ_1 and λ_2 two positive real numbers.

c) Verify that $(\boldsymbol{a} \mid \boldsymbol{a}) = 0$ implies $\boldsymbol{a} = \boldsymbol{0}$ iff the vector space is two-dimensional.

Problem 2.15. **Inner products for polynomials**

Let $\boldsymbol{p} = \left[\sum_{i=0}^{D} p_i\, x^i\right]$ and $\boldsymbol{q} = \left[\sum_{i=0}^{D} q_i\, x^i\right]$ be elements of the vector space of N-dimensional polynomials. Verify that the following rules define inner products on this space.

a) $\langle \boldsymbol{p} \mid \boldsymbol{q} \rangle = \sum_{i=0}^{N} \bar{p}_i\, q_i$

b) $\langle \boldsymbol{p} \mid \boldsymbol{q} \rangle_{[a,b]} = \int_a^b \mathrm{d}x \overline{\left[\sum_{i=0}^{D} p_i\, x^i\right]} \left[\sum_{i=0}^{D} q_i\, x^i\right]$ for $a < b \in \mathbb{R}$

c) Show that $\boldsymbol{p} = [1]$ and $\boldsymbol{q} = [x]$ are orthogonal with respect to the inner product defined in a). Under which condition are they also orthogonal for the inner product defined in b)?

2.8 Cartesian Coordinates

Theorem 2.2 entails an extremely elegant possibility to deal with vectors. We first illustrate the idea based on a two-dimensional example, Figure 2.14, and then we develop the general theory:

Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be two orthogonal vectors that have unit length,

$$\langle \boldsymbol{e}_1 | \boldsymbol{e}_1 \rangle = \langle \boldsymbol{e}_2 \mid \boldsymbol{e}_2 \rangle = 1 \quad \text{and} \quad \langle \boldsymbol{e}_1 | \boldsymbol{e}_2 \rangle = 0$$

For every vector $\boldsymbol{c}$ in the plane described by these two vectors, we can then find two numbers $c_1^{(e)}$ and $c_2^{(e)}$ such that

$$\boldsymbol{c} = c_1^{(e)}\, \boldsymbol{e}_1 + c_2^{(e)}\, \boldsymbol{e}_2$$

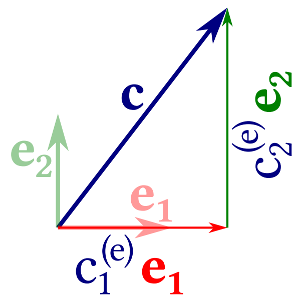


Figure 2.14: Representation of the vector $\boldsymbol{c}$ in terms of the orthogonal unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

Now the choice of the vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ entails that triangle with edge $\boldsymbol{c}$, $c_1^{(e)}\, \boldsymbol{e}_1$, and $c_2^{(e)}\, \boldsymbol{e}_2$ is right-angled and that

$$\begin{aligned} & c_i^{(e)} = |\boldsymbol{c}| \, \cos\angle(\boldsymbol{c}, \boldsymbol{e}_i) = \langle \boldsymbol{c} \mid \boldsymbol{e}_i \rangle \quad \text{for} \quad i \in \{1, 2\} \\ \Rightarrow \quad & \boldsymbol{c} = \langle \boldsymbol{c} \mid \boldsymbol{e}_1 \rangle\, \boldsymbol{e}_1 + \langle \boldsymbol{c} \mid \boldsymbol{e}_2 \rangle\, \boldsymbol{e}_2 \end{aligned}$$

This strategy to represent vectors applies in all dimensions.

Definition 2.12: Basis and Coordinates

Let $\mathsf{B} = \{\boldsymbol{e}_i,\ i \in \{1, \ldots, D\}$ be a set of D pairwise orthogonal unit vectors

$$\forall i, j \in \{1, \ldots, D\} : \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

in a vector space $(\mathsf{V}, \mathbb{F}, +, \cdot)$ with inner product $\langle \_ | \_ \rangle$. We say that B forms a *basis* for a D-dimensional vector space iff

$$\forall \boldsymbol{v} \in \mathsf{V}\ \exists v_i,\ i \in \{1, \ldots, D\} : \quad \boldsymbol{v} = \sum_{i=1}^{D} v_i^{(e)}\, \boldsymbol{e}_i$$

In that case we also have $v_i^{(e)} = \langle \boldsymbol{v} \mid \boldsymbol{e}_i \rangle$, $i \in \{1, \ldots, D\}$

and these numbers are called the *coordinates* of the vector $\boldsymbol{v}$. The number of vectors D in the basis of the vector space is denoted as *dimension of the vector space*.

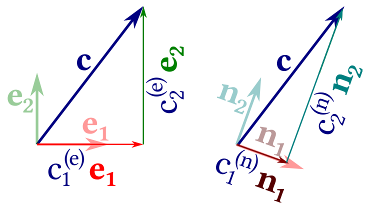


Figure 2.15: Representation of the vector $\boldsymbol{c}$ of Figure 2.14 in terms of the bases $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$.

Remark 2.19. The choice of a basis, and hence also of the coordinates, is not unique. Figure 2.15 shows the representation of a vector in terms of two different bases $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$. We suppress the superscript that indicates the basis when the choice of the basis is clear from the context.

Remark 2.20. For a given basis the representation in terms of coordinates is unique.

Proof. 1. The coordinates a_i of a vector $\boldsymbol{a}$ are explicitly given by $a_i = \langle a \mid \boldsymbol{e}_i \rangle$. This provides unique numbers for a given basis set.

2. Assume now that two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same coordinate representation. Then the vector-space properties imply

$$\left.\begin{aligned} \boldsymbol{a} &= \textstyle\sum_i c_i\, \boldsymbol{e}_i \\ \boldsymbol{b} &= \textstyle\sum_i c_i\, \boldsymbol{e}_i \end{aligned}\right\} \Rightarrow \boldsymbol{a} - \boldsymbol{b} = \left(\sum_i c_i\, \boldsymbol{e}_i\right) - \left(\sum_i c_i\, \boldsymbol{e}_i\right)$$
$$= \sum_i (c_i - c_i)\, \boldsymbol{e}_i = \sum_i 0\, \boldsymbol{e}_i = \mathbf{0}$$
$$\Rightarrow \quad \boldsymbol{a} = \boldsymbol{b}$$

Hence, they must be identical. □

Remark 2.21 (Kronecker δ_{ij}). It is convenient to introduce the abbreviation δ_{ij} for

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

where i, j are elements of some index set. This symbol is denoted as *Kronecker* δ. With the Kronecker symbol the condition on orthogonal unit vectors of a basis can more concisely be written as

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

Moreover, for $i, j \in \{1, \ldots, D\}$ the entries, δ_{ij}, describe a $D \times D$ matrix which is the neutral element for multiplication with another $D \times D$ matrix, and also with a vector of $\mathbb{R}^D$, when it is interpreted as a $D \times 1$ matrix.

Theorem 2.3: Scalar product on $\mathbb{R}^D$

The axioms of vector spaces and the inner product imply that

$$\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \sum_{i=1}^{D} \langle \boldsymbol{a} \mid i \rangle \, \langle i \mid \boldsymbol{b} \rangle = \sum_{i=1}^{D} a_i \bar{b}_i$$

Proof. We use the representation of $\boldsymbol{a} = \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \boldsymbol{e}_i$ and

$\boldsymbol{b} = \sum_j \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle\, \boldsymbol{e}_j$, and work step by step from the left to the aspired result

$$\begin{aligned}
\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle &= \left\langle \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \boldsymbol{e}_i \;\middle|\; \sum_j \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle\, \boldsymbol{e}_j \right\rangle \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \left\langle \boldsymbol{e}_i \;\middle|\; \sum_j \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle\, \boldsymbol{e}_j \right\rangle \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \sum_j \overline{\langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle}\, \langle \boldsymbol{e}_i \mid \boldsymbol{e}_j \rangle \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \sum_j \langle \boldsymbol{e}_j \mid \boldsymbol{b} \rangle\, \delta_{ij} \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle
\end{aligned}$$

and due to $a_i = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle$ and $\bar{b}_i = \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle$ we also have

$$\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \sum_i a_i\, \bar{b}_i$$

□

Remark 2.22. Einstein pointed out that the sums over pairs of identical indices arise ubiquitously in calculations like to proof of Theorem 2.3. He therefore adopted the convention that one always sums over pairs of identical indices, and does no longer explicitly write that down. This leads to substantially clearer representation of the calculation. For instance, the proof looks then as follows:

$$\begin{aligned}
\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle &= \left\langle \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \boldsymbol{e}_i \;\middle|\; \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle\, \boldsymbol{e}_j \right\rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \left\langle \boldsymbol{e}_i \;\middle|\; \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle\, \boldsymbol{e}_j \right\rangle \\
&= \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \overline{\langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle}\, \langle \boldsymbol{e}_i \mid \boldsymbol{e}_j \rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \overline{\langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle}\, \delta_{ij} \\
&= \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle\, \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle \\
\Rightarrow \quad \langle \boldsymbol{a} \mid \boldsymbol{b} \rangle &= a_i\, \bar{b}_i
\end{aligned}$$

Remark 2.23. Dirac pointed out that the vector product $\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle$ takes the form of the multiplication of a $D \times 1$ matrix for $\boldsymbol{a}$ and a $1 \times D$ matrix for $\boldsymbol{b}$. He suggested to symbolically write down these vectors as a *bra vector* $\langle a|$ and a *ket vector* $|b\rangle$. When put together as a bra-(c)-ket $\langle a|b\rangle$ one recovers the inner product, and introducing $|\boldsymbol{e}_i\rangle\langle \boldsymbol{e}_i|$ and observing observing Einstein notation comes down to inserting a unit matrix. For instance for 2×2 vectors

$$\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = (a_1, a_2) \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = (a_1, a_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle$$

Conceptually this is a very useful observation because it provides an easy rule to sort out what changes in the equations when one represents a problem in terms of a different basis.

Example 2.20: Changing coordinates from basis (e_i) to basis (n_i)

We observe Dirac's observation that the expressions $|e_i\rangle\langle e_i|$ and $|n_i\rangle\langle n_i|$ sandwiched between a bra and a ket amounts to multiplication with one. Hence, the coordinates change according to

$$a_i^{(n)} = \langle \boldsymbol{a} \mid \boldsymbol{n}_i \rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_j \rangle \langle \boldsymbol{e}_j | \boldsymbol{n}_i \rangle = a_j^{(e)} \langle \boldsymbol{e}_j | \boldsymbol{n}_i \rangle$$

which amounts to multiplying the vector with entries $(a_j^{(e)}, j = 1, \ldots, D)$ with the $D \times D$ matrix T with entries $t_{ji} = \langle \boldsymbol{e}_j | \boldsymbol{n}_i \rangle$.
On the other hand, for the inner products we have

$$\begin{aligned}
a_i^{(e)} \bar{b}_i^{(e)} &= \langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle \\
&= \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \langle \boldsymbol{n}_j \mid \boldsymbol{e}_i \rangle \langle \boldsymbol{e}_i \mid \boldsymbol{n}_k \rangle \langle \boldsymbol{n}_k | \boldsymbol{b} \rangle = \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \langle \boldsymbol{n}_j \mid \boldsymbol{n}_k \rangle \langle \boldsymbol{n}_k | \boldsymbol{b} \rangle \\
&= \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \, \delta_{jk} \, \langle \boldsymbol{n}_k | \boldsymbol{b} \rangle = \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \langle \boldsymbol{n}_j | \boldsymbol{b} \rangle = a_i^{(n)} \bar{b}_i^{(n)}
\end{aligned}$$

Its value does not change, even though the coordinates take entirely different values.

2.8.1 *Self Test*

Problem 2.16. **Cartesian Coordinates in the plane**

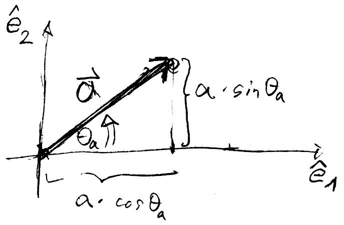
a) Mark the following points in a Cartesian coordinate system:

$$(0, 0) \quad (0, 3) \quad (2, 5) \quad (4, 3) \quad (4, 0)$$

Add the points (0, 0) (4, 3) (0, 3) (4, 0), and connect the points in the given order. What do you see?

b) What do you find when drawing a line segment connecting the following points?

$$(0, 0) \quad (1, 4) \quad (2, 0) \quad (-1, 3) \quad (3, 3) \quad (0, 0)$$



Problem 2.17. **Geometric and algebraic form of the scalar product** The sketch in the margin shows a vector $\boldsymbol{a}$ in the plane, and its representation as a linear combination of two orthonormal vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$,

$$\boldsymbol{a} = a \, \cos\theta_a \, \hat{\boldsymbol{e}}_1 + a \, \sin\theta_a \, \hat{\boldsymbol{e}}_2$$

Here, a is the length of the vector $\boldsymbol{a}$,
and $\theta_1 = \angle(\hat{\boldsymbol{e}}_1, \boldsymbol{a})$.

a) Analogously to $\boldsymbol{a}$ we consider another vector $\boldsymbol{b}$ with a representation

$$\boldsymbol{b} = b \, \cos\theta_b \, \hat{\boldsymbol{e}}_1 + b \, \sin\theta_b \, \hat{\boldsymbol{e}}_2$$

Employ the rules of scalar products, vector addition and multiplication with scalars to show that

$$\boldsymbol{a} \cdot \boldsymbol{b} = a\, b\, \cos(\theta_a - \theta_b)$$

Hint: Work backwards, expressing $\cos(\theta_a - \theta_b)$ in terms of $\cos\theta_a$, $\cos\theta_b$, $\sin\theta_a$, and $\sin\theta_b$.

b) As a shortcut to the explicit calculation of a) one can introduce the coordinates $a_1 = a \cos\theta_a$ and $a_2 = a \sin\theta_a$, and write $\boldsymbol{a}$ as a tuple of two numbers. Proceeding analogously for $\boldsymbol{b}$ one obtains

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

How does the product $\boldsymbol{a} \cdot \boldsymbol{b}$ look like in terms of these coordinates?

c) How do the arguments in a) and b) change for D dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors $\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_D$?

What changes when the basis is not orthonormal?
What if it is not even orthogonal?

Problem 2.18. **Pauli matrices form a basis for a 4D vector space**
Show that the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of the real vector space of 2×2 Hermitian matrices, $\mathbb{H}$, with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{H} \quad \Leftrightarrow \quad a_{ij} \in \mathbb{C} \ \wedge \ a_{ij} = a_{ji}^*$$

Show to that end

a) The matrices $\sigma_0, \dots, \sigma_4$ are linearly independent.

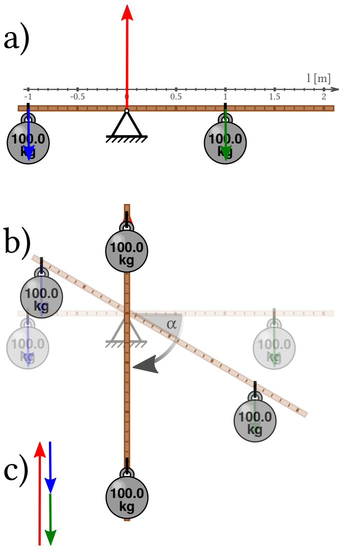
b) $x_0, \dots, x_4 \in \mathbb{R} \quad \Rightarrow \quad \sum_{i=0}^{4} x_i\, \sigma_i \in \mathbb{H}$

c) $M \in \mathbb{H} \quad \Rightarrow \quad \exists x_0, \dots, x_4 \in \mathbb{R} : M = \sum_{i=0}^{4} x_i\, \sigma_i$

What about linear combinations with coefficients $z_1, \dots, z_4$? Is $\sum_{i=0}^{4} z_i\, \sigma_i$ Hermitian? Do these matrices form a vector space?

2.9 Torques and Cross Products

adapted from rachaelvoorhees from arlington, va / wikimedia CC BY 2.0



based on from Jahobr/wikimedia CC0 1.0
Figure 2.16: a) The lever is balanced when two equal masses are attached at the same distance from the fulcrum. b) It is at (stable) rest only in a single position when equal weighs are attached at different distances. c) In all positions the sum of the forces on the beam, by the fulcrum and by the two weights, add to zero.

The pictures in the margin show the sign of a seesaw, a playground toy that works even for people with vastly different wight and size. Figure 2.16a) shows a balanced scale. When the forces acting on the scale do not add up to zero, we pick up the scale. It moves. The according force balance for the beam of the scale is shown in Figure 2.16c). In general the beam does not stay at rest, when the two masses are not attached at the same distance from the fulcrum. The force balance, Figure 2.16c), still hods, and the beam turns, rather than being lifted. The sum of attached forces tells us if an object is displaced. In analogy we introduce the *torque* to describe whether it turns.

When the beam is vertical there is no torque, and it takes its maximum when the beam is horizontal. In the former case the forces act parallel to the beam, and in the latter they act in orthogonal direction. Moreover, a weight that is attached at a larger distance to the fulcrum induces a larger torque, and the torque also increases with mass. This is expressed in the lever rule.

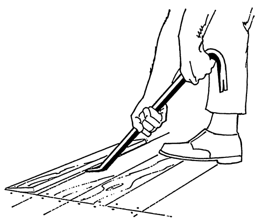
Example 2.21: Torques on a Lever

The torque T exerted by a lever is given by the product, $T = l\,F$, of the modulus of the force F acting vertical to the lever and the distance l between the fulcrum and the point where the force is applied, which is called *length of the lever arm*.

When several forces act on the same lever, then the total torque amounts to the sum of the torques induced by the individual forces, $T = \sum_i l_i\, F_i$. For the scale in Figure 2.16a) and b) we find

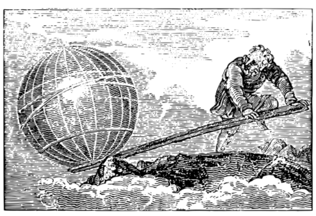
$$\begin{aligned}
T_a &= (1\,\mathrm{m})\;(100\,\mathrm{kg})\;(-g) + (-1\,\mathrm{m})\;(100\,\mathrm{kg})\;(-g) = 0 \\
T_b &= (1.5\,\mathrm{m})\;(100\,\mathrm{kg})\;(-g)\;\sin\alpha + (-1\,\mathrm{m})\;(100\,\mathrm{kg})\;(-g)\;\sin\alpha \\
&\quad \simeq -500 \sin\alpha\;\mathrm{kg\,m^2/s^2}
\end{aligned}$$

The torque vanishes only when $\alpha = 0$, as shown in the figure, and for the unstable tipping point $\alpha = \pi$.



Pearson Scott Foresman / Public domain
Figure 2.17: Action of a crowbar.

Remark 2.24. Adopting a lever where force is applied on a long arm allows one to move very heavy objects or break very stable objects. Common technological applications are the crowbar and the lever. Archimedes was so impressed by this principle that he is quoted to have remarked “Δοσ μοι που στω και κινω την γην” (Archimedes, 1878), i.e. “Give me but one firm spot on which to stand, and I will move the earth” (Oxford Dictionary of Quotations, 1953)



Mechanic’s Magazine cover of Vol II, Knight & Lacey, London, 1824./wikipedia, public domain
Figure 2.18: Illustration of Archimedes’ remark about moving the earth.

Observe the sign of the torque: In Example 2.21 it is positive for counterclockwise motion, and negative for clockwise motion. The axis of rotation is fixed by the fulcrum. However, when acting the crowbar, one applies a horizontal force to get the crowbar under the obstacle. This induces a rotation around a vertical axis. Subsequently, a vertical force is applied to lift the obstacle. It induces a rotation around a horizontal axes. The relation between the directions of the lever arm, the force, and the rotation axis is commonly illustrated by the right-hand rule (Figure 2.19): Here the arm points in the direction of the lever arm, the fingers in the direction of the applied force, and the thump along the rotation axis. This suggests to define torque as a product of two vectors, the arm ℓ and the force $\boldsymbol{F}$ that provide the torque, $\boldsymbol{T}$, which is a vector of length $|\ell|\,|\boldsymbol{F}|\,\sin\angle\ell,\boldsymbol{F}$ in a direction normal to the plane defined by ℓ and $\boldsymbol{F}$. This operation, $\boldsymbol{T} = \ell \times \boldsymbol{F}$ defines the cross product. We explore its properties in a mathematical digression.

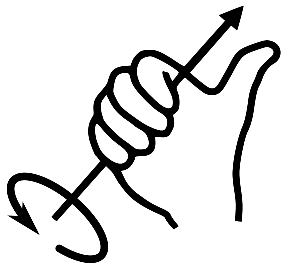


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Figure 2.19: Right-hand rule.

2.9.1 Algebraic properties of cross products

Definition 2.13: Cross product on $\mathbb{R}^3$

The *cross product* on the vector space $\mathbb{R}^3$ defines a binary operation $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

a) anticommutativity: $$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$$

b) distributivity: $$\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$$

c) compatibility with scalar product: $$(c\,\boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (c\,\boldsymbol{v}) = c\,(\boldsymbol{u} \times \boldsymbol{v})$$

d) symmetry of scalar triple product: $$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$$

Moreover for every right-handed set of three orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ we require

e) normalization: $$\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3) = 1$$

Remark 2.25. The cross product of a vector with itself vanishes

$$\forall \boldsymbol{v} \in \mathbb{R}^3 \;:\; \boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$$

Proof. Vanishing of $\boldsymbol{v} \times \boldsymbol{v}$ is a consequence of anticommutativity:

$$\boldsymbol{v} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{v} \quad \Rightarrow 2\,\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$$ □

Theorem 2.4: Right-handed orthonormal basis in $\mathbb{R}^3$

Let $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^3$ be orthonormal vectors, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \delta_{12}$. Then $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ form a right-handed orthonormal basis for $\mathbb{R}^3$, and we have

$$\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \begin{cases} 1 & \text{for} \quad ijk \in \{123, 231, 312\} \\ -1 & \text{for} \quad ijk \in \{132, 213, 321\} \\ 0 & \text{else} \end{cases}$$

Remark 2.26 (Levi-Civita tensor ε_{ijk}). It is convenient to introduce the abbreviation ε_{ijk} for

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for} \quad ijk \in \{123, 231, 312\} \\ -1 & \text{for} \quad ijk \in \{132, 213, 321\} \\ 0 & \text{else} \end{cases}$$

This symbol is denoted as *Levi-Civita tensor* ε_{ijk}. With this symbol the relations between right-handed orthogonal unit vectors of a basis can more concisely be written as

$$\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \varepsilon_{ijk}$$

Moreover, it immediately provides the following representation of the scalar triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ in terms of coordinates u_i, v_j, w_k, $i, j, k \in \{1, 2, 3\}$,

$$\left.\begin{array}{ll} \boldsymbol{u} & = \sum_{i=1}^{3} u_i\, \boldsymbol{e}_i \\ \boldsymbol{v} & = \sum_{j=1}^{3} v_j\, \boldsymbol{e}_j \\ \boldsymbol{w} & = \sum_{k=1}^{3} w_k\, \boldsymbol{e}_k \end{array}\right\} \quad \Rightarrow \quad \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, u_i\, v_j\, w_k$$

or even $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \varepsilon_{ijk}\, u_i\, v_j\, w_k$ with Einstein notation. The symmetry of the triple scalar product is an immediate consequence of the symmetry of the ε-tensor.

Proof. The identity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \varepsilon_{ijk}\, u_i\, v_j\, w_k$ follows from the compatibility with scalar product and the relation for the basis vectors $\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k)$. The details of the proof are given as Problem 2.19. □

Proof of Theorem 2.4. We show that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ form three orthonormal vectors. By assumption $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthonormal. Hence, we show that $\boldsymbol{e}_3$ is a unit vector that is orthogonal to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$:

$$\begin{aligned} \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 &= \boldsymbol{e}_1 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_2 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_1) = \boldsymbol{e}_2 \cdot \boldsymbol{0} = 0 \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 &= \boldsymbol{e}_2 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot \boldsymbol{0} = 0 \\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 &= \boldsymbol{e}_3 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3) = 1 \end{aligned}$$

□

Remark 2.27 (bac-cab rule). The double cross product can be expressed in terms of scalar products. Commonly this relation is stated in terms of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c} \in \mathbb{R}^3$,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})$$

and referred to as bac-cab rule.

Proof. We express the three vectors in terms of their coordinates with respect to the orthonormal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$,

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\, \boldsymbol{e}_i \quad \boldsymbol{b} = \sum_{j=1}^{3} b_j\, \boldsymbol{e}_j \quad \boldsymbol{c} = \sum_{k=1}^{3} c_k\, \boldsymbol{e}_k \quad \text{with} \quad a_i, b_j, c_k \in \mathbb{R}$$

and use the rules defining the cross products and inner products

$$
\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \left(\sum_{i=1}^{3} a_i\, \boldsymbol{e}_i\right) \times \left[\left(\sum_{j=1}^{3} b_j\, \boldsymbol{e}_j\right) \times \left(\sum_{k=1}^{3} c_k\, \boldsymbol{e}_k\right)\right] \\
&= \sum_{i,j,k=1}^{3} a_i\, b_j\, c_k \boldsymbol{e}_i \times (\boldsymbol{e}_j \times \boldsymbol{e}_k)
\end{aligned}
$$

When $j = k$ or when j and k are both different from i then the summand vanishes due to Remark 2.25. For $i = j \neq k$ one has $\boldsymbol{e}_i \times (\boldsymbol{e}_j \times \boldsymbol{e}_k) = -\boldsymbol{e}_k$, and for $i = k \neq j$ one has $\boldsymbol{e}_i \times (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \boldsymbol{e}_j$. Consequently,

$$
\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \sum_{i,k=1}^{3} a_i\, b_i\, c_k(-\boldsymbol{e}_k) + \sum_{i,j=1}^{3} a_i\, b_j\, c_i(\boldsymbol{e}_j) \\
&= \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})
\end{aligned}
$$

□

Remark 2.28 (Jacobi identity). The cross product obeys the Jacobi identity:

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} \times \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{0}$$

Proof. This can be verified by evaluating the triple cross products by the bac-cab rule. Details are give as Problem 2.20. □

Remark 2.29. In coordinate notation the cross product takes the form

$$
\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_2 b_3 - a_3 b_2 \end{pmatrix}
$$

Proof. For component k of $\boldsymbol{a} \times \boldsymbol{b}$ we have

$$
\begin{aligned}
[\boldsymbol{a} \times \boldsymbol{b}]_k &= \left[\left(\sum_{i=1}^{3} a_i\, \boldsymbol{e}_i\right) \times \left(\sum_{j=1}^{3} b_j\, \boldsymbol{e}_j\right)\right]_k = \sum_{i,j=1}^{3} a_i\, b_j\, \left[\boldsymbol{e}_i \times \boldsymbol{e}_j\right]_k \\
&= \sum_{i,j=1}^{3} a_i\, b_j\, \varepsilon_{ijk}
\end{aligned}
$$

In the remark this is explicitly written out for $k \in \{1, 2, 3\}$. □

2.9.2 *Geometric interpretation of cross products*

The cross product and the scalar triple product have distinct geometrical interpretations. The geometric meaning of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ can best be seen by adopting a basis where the first basis vector is parallel to $\boldsymbol{e}_1 = \boldsymbol{a}/|\boldsymbol{a}|$, and the second basis vector $\boldsymbol{e}_2$ lies orthogonal to $\boldsymbol{e}_1$ in the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$. The third basis vector will then be $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and hence also of $\boldsymbol{e}_1$ and $\boldsymbol{b}$ is denoted as θ. Thus, $\boldsymbol{b}$ can be written as $\boldsymbol{b} = b_1\, \boldsymbol{e}_1 + b_2\, \boldsymbol{e}_2 = |\boldsymbol{b}|\, (\cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2)$, cf. Figure 2.20). For this choice of the basis we find

$$
\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= |\boldsymbol{a}|\, \boldsymbol{e}_1 \times (b_1\, \boldsymbol{e}_1 + b_2\, \boldsymbol{e}_2) = |\boldsymbol{a}|\, b_1\, \boldsymbol{e}_1 \times \boldsymbol{e}_1 + |\boldsymbol{a}|\, b_2\, \boldsymbol{e}_1 \times \boldsymbol{e}_2 \\
&= |\boldsymbol{a}|\, |\boldsymbol{b}|\, \sin\theta\, \boldsymbol{e}_3
\end{aligned}
$$

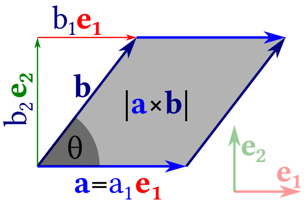


Figure 2.20: Geometric interpretation of the absolute value of the cross product.

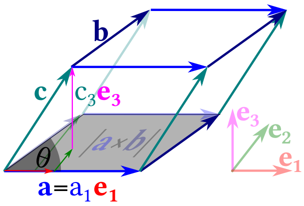


Figure 2.21: Geometric interpretation of the scalar triple product.

Figure 2.20 illustrates that $|\boldsymbol{a}|\,|\boldsymbol{b}|\,\sin\theta$ amounts to the area of the parallelogram spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence, the cross product amounts to a vector that is aligned vertically on the parallelogram, with a length that amounts to the area of the parallelogram.

In order to evaluate also the product $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ we introduce the coordinate representation of $\boldsymbol{c}$ as $\boldsymbol{c} = c_1\,\boldsymbol{e}_1 + c_2\,\boldsymbol{e}_2 + c_3\,\boldsymbol{e}_3$ (Figure 2.21), and observe

$$\begin{aligned}(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} &= |\boldsymbol{a} \times \boldsymbol{b}|\,\boldsymbol{e}_3 \cdot (c_1\,\boldsymbol{e}_1 + c_2\,\boldsymbol{e}_2 + c_3\,\boldsymbol{e}_3)\\ &= |\boldsymbol{a} \times \boldsymbol{b}|\,c_3 = a_1\,b_2\,c_3\end{aligned}$$

This amounts to the product of the area of the parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ multiplied by the height of the parallelepiped spanned by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Due to the special choice of the basis this volume amount to $a_1\,b_2\,c_3$ because all other contributions to the general expression $\sum_{ijk} a_i\,b_j\,c_k\,\varepsilon_{ijk}$ vanish. The symmetry of the scalar triple product, property d) in Definition 2.13, is understood from this perspective as the statement that the volume of the parallelepiped is invariant under (cyclic) renaming of the vectors that define its edges.

As a final remark, we emphasize that the geometric interpretation that we have given to the cross product holds in general — in spite of the special basis adopted in the derivation. It is a distinguishing feature of vector spaces that the scalar numbers that are derived from vectors take the same values every choice of the basis. It is up to the physicist to find the basis that admits the easiest calculations.

2.9.3 *The Torque*

The cross product equips us with the mathematical notions to define the torque on a body.

Definition 2.14: Torque

The *torque* $\boldsymbol{T}$ defines a force that is going to rotate a body around a position $\boldsymbol{q}_0$. Let $\boldsymbol{F}_i$ be the forces that attach the body at the positions $\boldsymbol{q}_i$ with respect to the considered origin. Then the torque is defined as

$$\boldsymbol{T} = \sum_i (\boldsymbol{q}_i - \boldsymbol{q}_0) \times \boldsymbol{F}_i$$

Remark 2.30. The value of the torque depends on the choice of the reference position $\boldsymbol{q}_0$.

Remark 2.31. In general, the torques induced by different forces point in different directions. They are added as vectors. We will further discuss this below in Example 2.22.

Axiom 2.2: Torque balance

Let N forces $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_N$ attack a body at the (body-fixed) positions $\boldsymbol{q}_i$. The body does not rotate around the position $\boldsymbol{q}_0$ as long as the sum of the torques induced by the forces add to zero, i.e. iff $\boldsymbol{0} = \boldsymbol{T} = \sum_{i=1}^{N} (\boldsymbol{q}_i - \boldsymbol{q}_0) \times \boldsymbol{F}_i$.

Example 2.22: Sailing boat

When a sailboat is going broad reach, as shown in Figure 2.22, the following forces are acting on the boat:

a) the wind in the sails generates a torque towards the bow around a horizontal axis that lies diagonal to the boat axis

b) the buoyancy of the water generates a torque along a horizontal axis parallel to the boat the counteracts heeling

c) the water drag on the hull generates a torque towards the bow around a horizontal axis that is orthogonal to the boat axis

d) the fin and the rudder generate lift forces that generate a torque around a vertical axis

e) the sailor stacks out in the trapeze to generate an additional torque in order to balance the torques

His aim is to minimize the heeling of the boat and to mximize the speed. The boat capsizes if he does not manage to balance the torques.

Gwicke commonswiki, public domain
Figure 2.22: A sailor stacking out in a trapeze in order to minimize the heeling of his sailboat.

2.9.4 Self Test

Problem 2.19. Fill in the details of the proof for Remark 2.26.

Problem 2.20. Fill in the details of the proof for Remark 2.28.

Problem 2.21. **Turning a wheel**
Two forces of magnitude 4 N are acting on a wheel of radius r that can freely rotate around its axis. What magnitude should a third force, $\boldsymbol{F}$, have that is attacking at a distance $r/2$ from the axis, such that there is no net torque acting on the wheel?

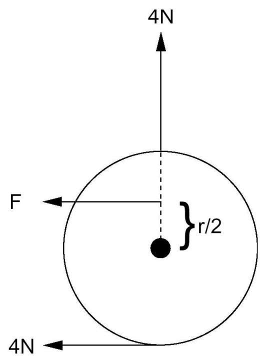
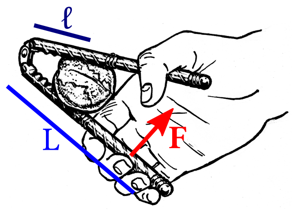


Figure 2.23: Setup for Problem 2.21.

Problem 2.22. **Nutcrackers**

A common type of nutcrackers employs the principle of lever arms to crack nuts with a reasonable amount of force (see Problem 2.22). We idealize the nut as a spring with spring constant $k = 1\,\text{kN/mm}$ and assume that it breaks when it is compressed by



based on Pearson Scott Foresman nutcracker-tool, public domain
Figure 2.24: Setup for Problem 2.22.

$\Delta = 0.6\,\text{mm}$. The nut is mounted at a distance of $l = 3\,\text{cm}$ from the joint of the nutcracker and the hand exerts a force $\boldsymbol{F}$ at a distance L.

a) Demonstrate that a force of magnitude $F = \dfrac{lk\Delta}{L}$ is required to crack the nut.

b) Calculate the numerical value of F.

c) If you try to crack the nut by placing it under a heavy stone: which mass should that stone have in order to crack the nut?

2.10 *Worked Example: the Mobile*

We describe here the setup of a traditional mobile where beams are supported by a string in the middle and balanced by attaching masses or further beams at their outer ends. The setup of a mobile can be layed out on a plane surface, as shown in Figure 2.25. The different partes of the mobile should not run into each other. Hence, they must not over overlap in the 2d layout.

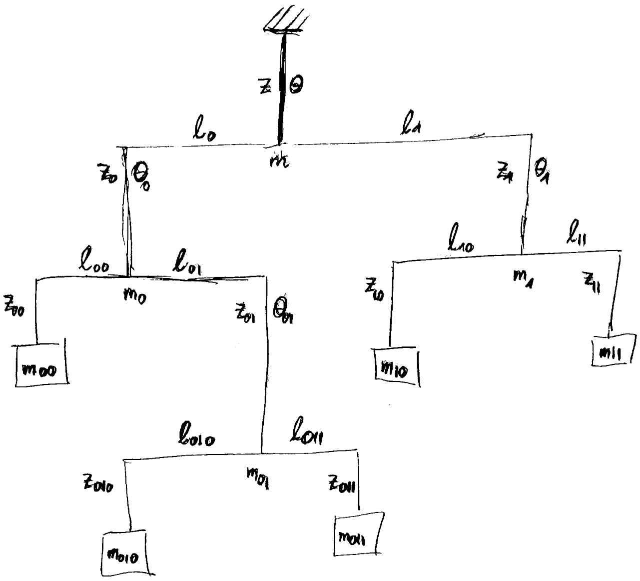


Figure 2.25: Notations for the mathematical description of the motion of a mobile. The mobile is supended at a string of length z that holds a beam with two sections of length ℓ_0 to the left and ℓ_1 to the right, respectively. The string holds the total mass m of the mobile. When suspended, the beam can rotate by an angle θ out of the plane.

The left arm of the uppermost beam has length ℓ_0, and it holds another beam with an overall additional mass m_0 that can take an angle θ_0 out of the plain in the suspended mobile. Similarly, the right arm has length ℓ_1, and it holds another beam with an overall additional mass m_1 that can take an out-of-plain angle θ_1. The situation further down is described by hirarchical binary indices, as indicated in the figure.

The mobile can represented as a binary tree. Each beam has two arms reaching left (0) and right (1). We assume that the mass of the beams may be neglected, and reach the masses at the far ends of the mobile, by going down from the suspension and marking the track by a sequence of 0 and 1. The leftmost mass, 00, of the mobile in Figure 2.25 is reached by going left, 0, twice. The next one in counterclockwise direction by going left 0, right 1, left 0, and hence denoted as 010, and so forth. Hence, the mobile is build of beams that are labeled by some index I. They support a total mass m_I, and can rotated out of the plane by an angle θ_I. The beam has two arms of length ℓ_{I0} to the left and ℓ_{I1} to the right that support masses m_{I0}

and m_{I1} attached to strings of length z_{I0} and z_{I1}. This hirarchichal setup of the descriptions allows us to reduce the requirement of stability by a condition that the forces and theorques acting on the beams must be balanced. For the forces this implies

$$\boldsymbol{F}_I = \boldsymbol{F}_{I0} + \boldsymbol{F}_{I1} \quad \Rightarrow \quad m_I = m_{I0} + m_{I1}$$

and for the torques we find

$$\ell_{I0} m_{I0} g = \ell_{I1} m_{I1} g \quad \Rightarrow \quad \ell_{I0} m_{I0} = \ell_{I1} m_{I1}$$

When we take all masses to take the same value m in Figure 2.25, we hence find

$$\ell_{010} = \ell_{011} \qquad \ell_{10} = \ell_{11} \qquad \ell_{00} = 2\,\ell_{01} \qquad 3\,\ell_0 = 2\,\ell_1$$

Moreover, vector calculus provides an effective means to specify the positions of the masses. We select the support of the mobile as origin of the coordinate system. The support of the uppermost beam ist at position $(0, 0, -z)$. Then the far ends of the uppermost beam are at positions $\boldsymbol{l}_o = (-\ell_0 \cos\theta, -\ell_0 \sin\theta, -z)$ and $\boldsymbol{l}_1 = (\ell_1 \cos\theta, -\ell_1 \sin\theta, -z)$, respectively. Moreover, from the left end we reach the far ends of the next beam by the displacement vectors $\boldsymbol{l}_{00} = (-\ell_{00} \cos\theta_0, -\ell_{00} \sin\theta_0, -z_0)$ and $\boldsymbol{l}_{01} = (\ell_{01} \cos\theta_0, \ell_{01} \sin\theta_0, -z_0)$. Hence, the positions of the first two masses can be represented by the following sums of vectors

$$\boldsymbol{q}_{00} = \boldsymbol{l}_0 + \boldsymbol{l}_{00} - \begin{pmatrix} 0 \\ 0 \\ z_{00} \end{pmatrix} = \begin{pmatrix} -\ell_0 \cos\theta - \ell_{00} \cos\theta_0 \\ -\ell_0 \sin\theta - \ell_{00} \sin\theta_0 \\ -z - z_0 - z_{00} \end{pmatrix}$$

$$\boldsymbol{q}_{010} = \boldsymbol{l}_0 + \boldsymbol{l}_{01} + \boldsymbol{l}_{010} - \begin{pmatrix} 0 \\ 0 \\ z_{010} \end{pmatrix} = \begin{pmatrix} -\ell_0 \cos\theta + \ell_{01} \cos\theta_0 - \ell_{010} \cos\theta_{01} \\ -\ell_0 \sin\theta + \ell_{01} \sin\theta_0 - \ell_{010} \sin\theta_{01} \\ -z - z_0 - z_{01} - z_{010} \end{pmatrix}$$

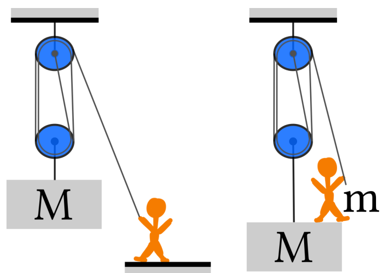
We urge the reader to also work out the expressions for the positions of the other masses.

add discussion and stability analysis for bended beams

2.11 Problems

2.11.1 Rehearsing Concepts

Problem 2.23. **Tackling tackles and pulling pulleys**



a) Which forces are required to hold the balance in the left and the right sketch?

b) Let the sketched person and the weight have masses of $m = 75\,\text{kg}$ and $M = 300\,\text{kg}$, respectively. Which power is required then to haul the line at a speed of $1\,\text{m/s}$.
Hint: The power is defined here as the change of) $M g\, z(t)$ and $(M + m)\, g\, z(t)$, per unit time, respectively. Verify by dimensional analysis that this is a meaningful definition.

2.11.2 Practicing Concepts

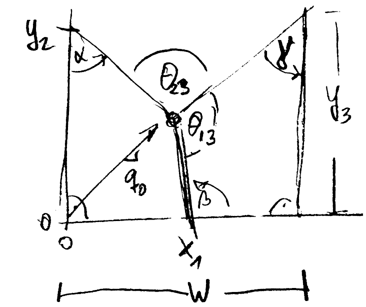


Figure 2.26: Setup of Problem 2.24.

Problem 2.24. **Angles between three balanced forces**

We consider three masses m_1, m_2, and m_3. With three ropes they are attached to a ring at position $\boldsymbol{q}_0$. The ropes with the attached masses hang over the edge of a table at the fixed positions $\boldsymbol{q}_1 = (x_1, 0)$, $\boldsymbol{q}_2 = (0, y_2)$, and $\boldsymbol{q}_3 = (w, y_3)$. Here, w denotes the width of the table board. We now determine the angles θ_{ij} between the ropes from $\boldsymbol{q}_0$ to $\boldsymbol{q}_i$ and $\boldsymbol{q}_j$, respectively.

a) Let $\hat{\boldsymbol{e}}_i = (\boldsymbol{q}_i - \boldsymbol{q}_0)/|\boldsymbol{q}_i - \boldsymbol{q}_0|$ be the unit vectors pointing from the ring to the positions where the ropes hang over the table edge, and θ_{ij} be the angle between $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}_j$. Argue why

$$\mathbf{0} = \sum_{i=1}^{3} m_i \, \hat{\boldsymbol{e}}_i$$

Multiplying this equation with $\hat{\boldsymbol{e}}_1, \dots \hat{\boldsymbol{e}}_3$ provides three equations that are linear in $\cos\theta_{ij}$. The first one is $0 = M_1 + M_2 \cos\theta_{12} + M_3 \cos\theta_{13}$. Find the other two equation, and solve the equations as follows.
From the equation that is given above you find $\cos\theta_{12}$ in terms of $\cos\theta_{13}$.
Inserting this into the other equation involving $\cos\theta_{12}$ (and rearranging terms) provides $\cos\theta_{23}$ in terms of $\cos\theta_{13}$.
Inserting this into the third equation provides

$$\cos\theta_{13} = \frac{M_2^2 - M_1^2 - M_3^2}{2\, M_1\, M_3}$$

b) Which angle θ_{23} do you find when $M_1 = M_2 = M_3$? The three forces have the same absolute value in this case. Which symmetry argument does then also provide the value of the angle?

c) Determine also the other two angles θ_{13} and θ_{12}. They can also be found from a symmetry argument without calculation.
Hint: The angles do not care which mass you denote as 1, 2, and 3.

d) Note that we found the angles θ_{ij} without referring to the positions $\boldsymbol{q}_1, \dots \boldsymbol{q}_3$! Make a sketch what this implies for the position of the ring, and how $\boldsymbol{q}_0$ changes qualitatively upon changing a mass.

The calculation of the position $\boldsymbol{q}_0$ can then be attacked by observing that

$$\boldsymbol{q}_0 = \boldsymbol{q}_1 + l_1 \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = \boldsymbol{q}_2 + l_2 \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} = \boldsymbol{q}_3 + l_3 \begin{pmatrix} -\sin\gamma \\ -\cos\gamma \end{pmatrix}$$

where l_i is the distance of the ring to the position where rope i hangs over the table. Further, the fact that the angles of quadrilaterals add to 2π provides

$$\alpha = \theta_{23} - \gamma \qquad \text{and} \qquad \beta = \frac{3\pi}{2} - \gamma - \theta_{13}$$

Altogether these are 8 equations to determine the two components of q_0, $l_1, \ldots l_3$, and the angles α, β, and γ. Determine q_0.

Problem 2.25. **Torques acting on a ladder**

The sketch in the margin shows the setup of a ladder leaning to the roof of a hut. The indicated angle from the downwards vertical to the ladder is denoted as θ. There is a gravitational force of magnitude Mg acting of a ladder of mass M. At the point where it leans to the roof there is a normal force of magnitude F_r acting from the roof to the ladder. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude γF_f. This is again the sketch to the ladder leaning to the roof of a hut. The angle from the downwards vertical to the ladder is denoted θ. There is a gravitational force of magnitude Mg acting of a ladder. At the point where it leans to the roof there is a normal force of magnitude F_r. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude F_f.

change to problem given on homework sheet 3.

a) In principle there also is a friction force $\gamma_r F_r$ acting at the contact from the ladder to the roof. Why is it admissible to neglect this force?
Remark: There are at least two good arguments.

b) Determine the vertical and horizontal force balance for the ladder. Is there a unique solution?

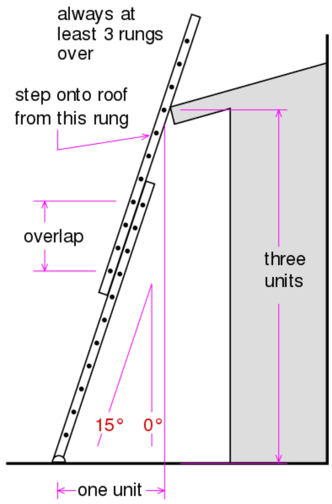
c) The feet of the ladder start sliding when F_f exceeds the maximum static friction force γF_g. What does this condition entail for the angle θ?
Assume that $\gamma \simeq 0.3$ What does this imply for the critical angle θ_c.

d) Where does the mass of the ladder enter the discussion? Do you see why?

e) Determine the torque acting on the ladder. Does it matter whether you consider the torque with respect to the contact point to the roof, the center of mass, or the foot of the ladder?

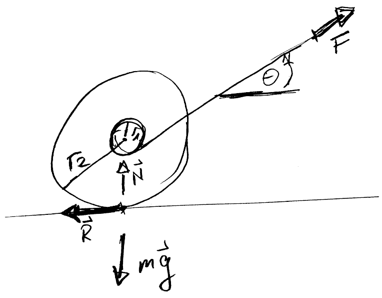
f) The ladder slides when the modulus of the friction force F_f exceeds a maximum value $\mu_S F_g$ where μ is the static friction coefficient for of the ladder feet on the ground. For metal feet on a wooden ground it takes a value of $\mu_S \simeq 2$. What does that tell about the angles where the ladder starts to slide?

g) Why does a ladder commonly starts sliding when when a man has climbed to the top? Is there anything one can do against it? Is that even true, or just an urban legend?



original: Bradley, vector: Sarang / wikimedia public domain
Figure 2.27: Setup for Problem 2.25: leaning a ladder to a roof.

Problem 2.26. **Walking a yoyo**

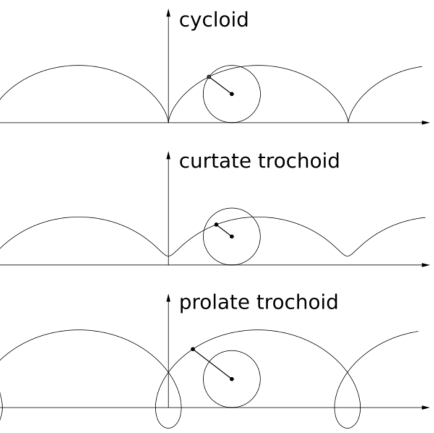


The sketch to the right shows a yoyo of mass m standing on the ground. It is held at a chord that extends to the top right. There are four forces acting on the yoyo: gravity $m\boldsymbol{g}$, a normal force $\boldsymbol{N}$ from the ground, a friction force $\boldsymbol{R}$ at the contact to the ground, and the force $\boldsymbol{F}$ due to the chord. The chord is wrapped around an axle of radius r_1. The outer radius of the yoyo is r_2.

a) Which conditions must hold such that there is no net force acting on the center of mass of the yoyo?

b) For which angle θ does the torque vanish?

c) Perform an experiment: What happens for larger and for smaller angels θ? How does the yoyo respond when fix the height where you keep the chord and pull continuously?



based on Kmhkmh Zykloiden, CC BY 4.0

Problem 2.27. **Retro-reflector paths on bike wheels.**

The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retro-reflectors fixed in the sparks enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio ρ of the reflectors distance d to the wheel axis and the wheel radius r. A small stone in the profile traces a cycloid ($\rho = 1$). Animations of the trajectories can be found at `https://en.wikipedia.org/wiki/Trochoid` and `http://katgym.by.lo-net2.de/c.wolfseher/web/zykloiden/zykloiden.html`.

A trochoid is most easily described in two steps: Let $\boldsymbol{M}(\theta)$ be the position of the center of the disk, and $\boldsymbol{D}(\theta)$ the vector from the center to the position $\boldsymbol{q}(\theta)$ that we follow (i.e. the position of the retro-reflector) such that $\boldsymbol{q}(\theta) = \boldsymbol{M}(\theta) + \boldsymbol{D}(\theta)$.

a) The point of contact of the wheel with the street at the initial time t_0 is the origin of the coordinate system. Moreover, we single out one spark and denote the change of its angle with respect to its initial position as θ. Note that negative angles θ describe forward motion of the wheel!

Sketch the setup and show that

$$\boldsymbol{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix}, \qquad \boldsymbol{D}(\theta) = \begin{pmatrix} -d\,\sin(\varphi+\theta) \\ d\,\cos(\varphi+\theta) \end{pmatrix}.$$

check signs of components of $\boldsymbol{D}$

What is the meaning of φ in this equation?

b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time, $L = \int_{t_0}^{t} \mathrm{d}t\ |\dot{\boldsymbol{q}}(\theta(t))|$.

Show that therefore

$$L = r \int_0^\theta \mathrm{d}\theta \, \sqrt{1 + \rho^2 + 2\rho \, \cos(\varphi + \theta)}$$

c) Consider now the case of a cycloid and use $\cos(2x) = \cos^2 x - \sin^2 x$ to show that the expression for L can then be written as

$$L = 2\, r \int_0^\theta \mathrm{d}\theta \, \left| \cos \frac{\varphi + \theta}{2} \right|$$

How long is one period of the track traced out by a stone picked up by the wheel profile?

2.11.3 Mathematical Foundation

Problem 2.28. **Linear Dependence of three vectors in 2D**

In the lecture I pointed out that every vector $\boldsymbol{v} = (v_1, v_2)$ of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{v} = \alpha \, \boldsymbol{a} + \beta \, \boldsymbol{b}$$

In this exercise we revisit this statement for $\mathbb{R}^2$ with the standard forms of vector addition and multiplication by scalars.

a) Provide a triple of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ such that $\boldsymbol{v}$ can *not* be represented as a scalar combination of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b) To be specific we henceforth fix

$$\boldsymbol{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} , \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad \boldsymbol{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers α and β such that

$$\boldsymbol{v} = \alpha \, \boldsymbol{a} + \beta \, \boldsymbol{b}$$

c) Consider now also a third vector

$$\boldsymbol{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for (α, β, γ) such that

$$\boldsymbol{v} = \alpha \, \boldsymbol{a} + \beta \, \boldsymbol{b} + \gamma \boldsymbol{c}$$

What is the general constraints on (α, β, γ) such that $\boldsymbol{v} = \alpha \, \boldsymbol{a} + \beta \, \boldsymbol{b} + \gamma \boldsymbol{c}$.
What does this imply on the number of solutions?

d) Discuss now the linear dependence of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ by exploring the solutions of

$$\mathbf{0} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b} + \gamma \boldsymbol{c}$$

How are the constraints for the null vector related to those obtained in part c)?

Problem 2.29. **Algebraic number fields**

Consider the set $\mathbb{K} = \mathbb{Q} + \mathrm{I}\mathbb{Q}$ with $\mathrm{I}^2 \in \mathbb{Q}$. We define the operations $+$ and $\cdot$ in analogy to those of the complex numbers (cf. Example 2.12): For $z_1 = x_1 + \mathrm{I}y_1$ and $z_2 = x_2 + \mathrm{I}y_2$ we have $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ and

$$\begin{aligned} \forall z_1, z_2 \in \mathbb{K} : & z_1 + z_2 = (x_1 + x_2) + \mathrm{I}\,(y_1 + y_2) \\ & z_1 \cdot z_2 = (x_1\, x_2 + \mathrm{I}^2\, y_1 y_2) + \mathrm{I}\,(x_1\, y_2 + y_1\, x_2) \\ \forall c \in \mathbb{Q}, z = (x + \mathrm{i}y) \in \mathbb{K} : & cz = c\, x + \mathrm{I}\, y \end{aligned}$$

a) Let I be a rational number, $\mathrm{I} \in \mathbb{Q}$. Show that $\mathbb{K} = \mathbb{Q}$.

b) Consider $\mathrm{I} = \sqrt{10}$. Show that $\mathbb{K}$ is a field that is different from $\mathbb{Q}$.

c) Consider $\mathrm{I} = \sqrt{8}$. In this case $\mathbb{K}$ is *not* a field! Why?

Find the general rule: For which natural numbers n does $\mathrm{I} = \sqrt{n}$ provide a non-trivial field?
Remark: Non-trivial means here different from $\mathbb{Q}$.

Problem 2.30. **Bases for polynomials**

We consider the set of polynomials $\mathbb{P}_N$ of degree N with real coefficients p_n, $n \in \{0, \dots, N\}$,

$$\mathbb{P}_N := \left\{ \boldsymbol{p} = \left(\sum_{k=0}^{N} p_n\, x^k \right) \quad \text{mit} \ \ p_n \in \mathbb{R}, n \in \{0, \dots, N\} \right\}$$

a) Demonstrate that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is a vector space when one adopts the operations

$$\forall \quad \boldsymbol{p} = \left(\sum_{k=0}^{N} p_n\, x^k \right) \in \mathbb{P}_N\,, \quad \boldsymbol{q} = \left(\sum_{k=0}^{N} q_n\, x^k \right) \in \mathbb{P}_N\,, \ \text{and} \ \ c \in \mathbb{R} \ :$$

$$\boldsymbol{p} + \boldsymbol{q} = \left(\sum_{k=0}^{N} (p_k + q_k)\, x^k \right) \qquad \text{and} \qquad c \cdot \boldsymbol{p} = \left(\sum_{k=0}^{N} (c\, p_k)\, x^k \right) .$$

(b) Demonstrate that

$$\boldsymbol{p} \cdot \boldsymbol{q} = \left(\int_0^1 \mathrm{d}x \left(\sum_{k=0}^{N} p_k\, x^k \right) \left(\sum_{j=0}^{N} q_j\, x^j \right) \right) ,$$

establishes a scalar product on this vector space.

(c) Demonstrate that the three polynomials $\boldsymbol{b}_0 = (1)$, $\boldsymbol{b}_1 = (x)$ and $\boldsymbol{b}_2 = (x^2)$ form a basis of the vector space $\mathbb{P}_2$: For each polynomial $\boldsymbol{p}$ in $\mathbb{P}_2$ there are real numbers x_k, $k \in \{0, 1, 2\}$, such that $\boldsymbol{p} = x_0\,\boldsymbol{b}_0 + x_1\,\boldsymbol{b}_1 + x_2\,\boldsymbol{b}_2$. However, in general we have $x_i \neq \boldsymbol{p} \cdot \boldsymbol{b}_i$. Why is that?
Hint: Is this an orthonormal basis?

(d) Demonstrate that the three vectors $\hat{\boldsymbol{e}}_0 = (1)$, $\hat{\boldsymbol{e}}_1 = \sqrt{3}\,(2\,x - 1)$ and $\hat{\boldsymbol{e}}_2 = \sqrt{5}\,(6\,x^2 - 6\,x + 1)$ are orthonormal.

(e) Demonstrate that every vector $\boldsymbol{p} \in \mathbb{P}_2$ can be written as a scalar combination of $(\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$,

$$\boldsymbol{p} = (\boldsymbol{p} \cdot \hat{\boldsymbol{e}}_0)\,\hat{\boldsymbol{e}}_0 + (\boldsymbol{p} \cdot \hat{\boldsymbol{e}}_1)\,\hat{\boldsymbol{e}}_1 + (\boldsymbol{p} \cdot \hat{\boldsymbol{e}}_2)\,\hat{\boldsymbol{e}}_2\,.$$

Hence, $(\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ form an orthonormal basis of $\mathbb{P}_2$.

*(f) Find a constant c and a vector $\hat{\boldsymbol{n}}_1$, such that $\hat{\boldsymbol{n}}_0 = (c\,x)$ and $\hat{\boldsymbol{n}}_1$ form an orthonormal basis of $\mathbb{P}_1$.

Problem 2.31. **Systems of linear equations**

A system of N linear equations of M variables $x_1, \ldots x_M$ comprises N equations of the form

$$\begin{aligned}
b_1 &= a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1M}\,x_M \\
b_2 &= a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2M}\,x_M \\
\vdots &\quad \vdots \\
b_N &= a_{N1}\,x_1 + a_{N2}\,x_2 + \cdots + a_{NM}\,x_M
\end{aligned}$$

where $b_i, a_{ij} \in \mathbb{R}$ for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$.

a) Demonstrate that the linear equations $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$ form a vector space when one adopts the operations

$$\begin{aligned}
\forall \quad & \boldsymbol{p} = \big[p_0 = p_1\,x_1 + p_2\,x_2 + \cdots + p_M\,x_M\big] \in \mathbb{L}_N\,, \\
& \boldsymbol{q} = \big[q_0 = q_1\,x_1 + q_2\,x_2 + \cdots + q_M\,x_M\big] \in \mathbb{L}_N\,, \\
& c \in \mathbb{R}\ :
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{p} + \boldsymbol{q} &= \big[p_0 + q_0 = (p_1 + q_1)\,x_1 + (p_2 + q_2)\,x_2 + \cdots + (p_M + q_M)\,x_M\big] \\
c \cdot \boldsymbol{p} &= \big[c\,p_0 = c\,p_1\,x_1 + c\,p_2\,x_2 + \cdots + c\,p_M\,x_M\big]\,.
\end{aligned}$$

How do these operations relate to the operations performed in Gauss elimination to solve the system of linear equations?

b) The system of linear equations can also be stated in the following form

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M$$

$$\Leftrightarrow \qquad \boldsymbol{b} = x_1\,\boldsymbol{a}_1 + x_2\,\boldsymbol{a}_2 + \cdots + x_M\,\boldsymbol{a}_M$$

where $\boldsymbol{b}$ is expressed as a linear combination of $\boldsymbol{a}_1, \dots \boldsymbol{a}_M$ by means of the numbers $x_1, \dots, x_M$. What do the conditions on linear independence and representation of vectors by means of a basis tell about the existence and uniqueness of the solutions of a system of linear equations.

2.11.4 *Transfer and Bonus Problems, Riddles*

Problem 2.32. **Crossing a River.**

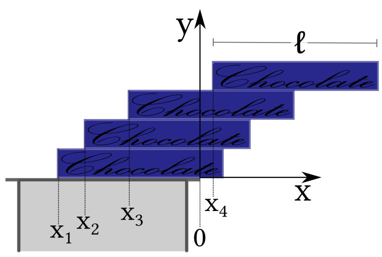
A ferry is towed at the bank of a river of width $B = 100$ m that is flowing at a velocity $v_F = 4$ m/s to the right. At time $t = 0$ s it departs and is heading with a constant velocity $v_B = 10$ km/h to the opposite bank.

a) When will it arrive at the other bank when it always heads straight to the other side? (In other words, at any time its velocity is perpendicular to the river bank.)

   How far will it drift downstream on its journey?

b) In which direction (i.e. angle of velocity relative to the downstream velocity of the river) must the ferryman head to reach exactly at the opposite side of the river?

   Determine first the general solution. What happens when you try to evaluate it for the given velocities?

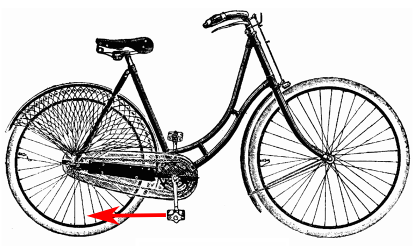


Problem 2.33. **Piling bricks**

At Easter and Christmas Germans consume enormous amounts of chocolate. If you happen to come across a considerable pile of chocolate bars (or beer mats, or books, or anything else of that form) I recommend the following experiment:

a) We consider N bars of length l piled on a table. What is the maximum amount that the topmost bar can reach beyond the edge of the table.

b) The sketch above shows the special case $N = 4$.
   However, what about the limit $N \to \infty$?



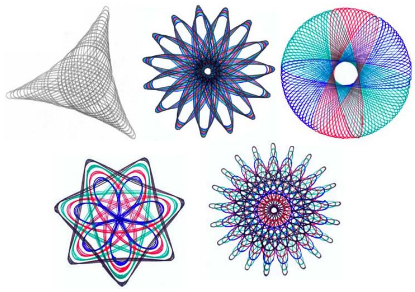
adapted from picture "Damenfahrrad von 1900" in article "Fahrrad" of Lueger (1926–1931)

Problem 2.34. **Where does the bike go?**

Consider the picture of the bicycle to the left. The red arrow indicates a force that is acting on the paddle in backward direction.
Will the bicycle move forwards or backwards?
Take a bike and do the experiment!

Problem 2.35. **Hypotrochoids, roulettes, and the Spirograph.**

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).


wikimedia, public domain

a) Consider the track of a pole attached to a disk with n cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the curves developed in Problem 2.27 such that you can describe hypotrochoids.

c) Test your result by writing a Python program that plots the curves for given m and n.

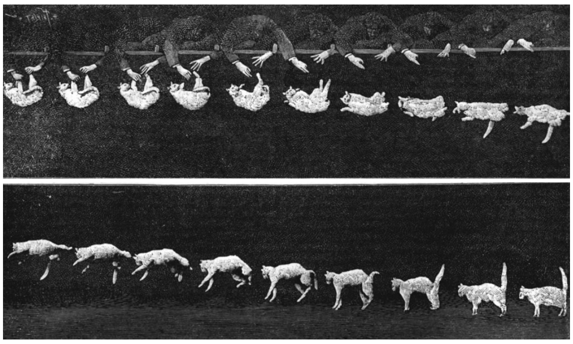
2.12 Further Reading

The second chapter of Großmann (2012) provides a clear and concise introduction to the mathematical framework of vectors with an emphasis on applications to physics problems.

A nice discussion of force and torque balances with many worked exercises can be found in Chapter 2 of Morin (2007).

3
Newton's Laws

In Chapter 2 we explored how several forces that act on a body can be subsumed into a net total force and torque. The body stays in rest, say at position $\boldsymbol{q}_0$, when the net force and torque vanish. Now we explore how the forces induce motion and how the position of the body evolves in time, $\boldsymbol{q}(t)$, when it is prepared with an initial condition $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ at the initial time t_0.



Photographs of a Tumbling Cat. Nature 51, 80–81 (1894)

At the end of this chapter we will be able to discuss the likelihood for injuries in different types of accidents, be it men or cat or mice. Why do the cats go away unharmed in most cases when they fall from a balcony, while an old professor should definitely avoid such a fall. As a worked example we will discuss water rockets.

3.1 Motivation and Outline: What is causing motion?

Every now and then I make the experience that I sit in a train, reading a book. Then I look out of the window, realize that we are passing a train, feeling happy that we are further approaching my final destination; and then I realize that the train is moving and my train is still in the station. Indeed, the motion of objects in my compartment is exactly identical, no matter whether it is at rest or moves with a constant velocity; be it zero in the station, at 15 m/s in a local commuter train, or 75 m/s in a Japanese high-speed train. However, changes of velocity matter. I forcefully experience the change of speed of the train during an emergency break, and coffee is spilled when it takes too sharp a turn.

Modern physics was born when Galileo and Newton formalized this experience by saying that bodies (e.g. the set of bodies in the compartment of a train) move in a straight line with a constant velocity as long as there is no net force acting on the bodies, and that the change of its velocity is proportional to the applied force.

Outline

In the first part of this chapter we will relate temporal changes of positions and velocities to time derivatives. Subsequently, we can formulate equations of motion that relate these changes to forces. The last part of the chapter deals with strategies to find solutions by making use of conservation laws.

mass	m
position	$\boldsymbol{q}(t)$
velocity	$\dot{\boldsymbol{q}}(t), \boldsymbol{v}(t)$
acceleration	$\ddot{\boldsymbol{q}}(t)$
forces	$\boldsymbol{F}_\alpha(\boldsymbol{q}, t)$

Table 3.1: Notations adopted to describe the motion of a particle. A single dot denotes the time derivative, and double dot the second derivative with respect to time.

3.2 *Time derivatives of vectors*

In this section we consider the motion of a particle with mass m that is at position $\boldsymbol{q}(t)$ at time t. Its average velocity $\boldsymbol{v}_{\text{av}}(t, \Delta t)$ during the time interval $[t, t + \Delta t]$ is

$$\boldsymbol{v}_{\text{av}}(t, \Delta t) = \frac{\boldsymbol{q}(t + \Delta t) - \boldsymbol{q}(t)}{\Delta t}$$

When the limit $\lim_{\Delta t \to 0} \boldsymbol{v}_{\text{av}}(t, \Delta t)$ exists[1] we can define the velocity of the particle at time t,

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}(t + \Delta t) - \boldsymbol{q}(t)}{\Delta t} \tag{3.2.1}$$

[1] The discussion of this limit for general functions is a core topic of vector calculus. For our present purpose the intuitive understanding based on the idea that $\boldsymbol{q}(t + \Delta t) \simeq \boldsymbol{q}(t) + \Delta t\ \boldsymbol{v}(t)$ provides the right idea. To provide a hint for the origin of the mathematical subtleties we point out that the approximation works unless there is an *instantaneous* collision with a wall at some point in the time interval $]t, t + \Delta t[$. In physics we try our luck, and fix the problem when we face it. Indeed, upon a close look there are no instantaneous collisions in physics, see Problem 3.15.

The velocity is then the time derivative of the position, and in an immediate generalization of the time derivative of scalar functions we also write

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t}$$

Finally, we point out that the components of the time derivative of a vector amount to the derivatives of the components.

Theorem 3.1: Time derivatives of vectors

Let $\boldsymbol{a}(t)$ be a vector with time-dependent components $a_i(t)$ with respect to orthonormal basis $\{\boldsymbol{e}_i, i = 1 \cdots D\}$ that is fixed in time.
Then $\dot{\boldsymbol{a}}(t) = \sum_i \dot{a}_i(t)\, \hat{\boldsymbol{e}}_i$, i.e. the components of $\dot{\boldsymbol{a}}(t)$ amount to the time derivatives of the components of $\boldsymbol{a}(t)$.

Proof. For each time we have $\boldsymbol{a}(t) = \sum_i a_i(t)\, \hat{\boldsymbol{e}}_i$ where it is understood that the sum runs over $i = 1 \cdots D$. We insert this into the definition, Equation (3.2.1), of the the time derivative and use the linearity of scalar products with vectors to obtain

$$\begin{aligned}
\dot{\boldsymbol{a}}(t) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{a}(t+\Delta t) - \boldsymbol{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sum_i a_i(t+\Delta t)\, \hat{\boldsymbol{e}}_i - \sum_i a_i(t)\, \hat{\boldsymbol{e}}_i}{\Delta t} \\
&= \lim_{\Delta t \to 0} \sum_i \hat{\boldsymbol{e}}_i \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} = \sum_i \hat{\boldsymbol{e}}_i \lim_{\Delta t \to 0} \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} \\
&= \sum_i \hat{\boldsymbol{e}}_i\, \dot{a}_i(t)
\end{aligned}$$

The subtle step here, from a mathematical point of view, is the swapping of the limit and the sum in the second line of the argument. Courses on vector calculus will spell out the assumptions needed to justify this step (or, more interestingly from a physics perspective, under which conditions it fails). □

The change of the velocity will be denoted as acceleration. Based on an analogous argument as for the velocity, it will be written as a time derivative

Definition 3.1: Acceleration

The time derivative of the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{q}}(t)$ is denoted as *acceleration*, and written as

$$\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{q}}(t)$$

In the next section it will be related to the action of forces $\boldsymbol{F}(\boldsymbol{q}, t)$ acting on a particle that resides at the position $\boldsymbol{q}$ at time t.

3.2.1 *Self Test*

Problem 3.1. **Derivatives of Elementary Functions**

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^x = \mathrm{e}^x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ln x = x^{-1}$$

Use only the three rules for derivatives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x) + g(x)\big) &= f'(x) + g'(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x) g(x)\big) &= f'(x)\, g(x) + f(x)\, g'(x) \\
\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) &= g'(x)\, f'\big(g(x)\big)
\end{aligned}$$

to work out the following derivatives

a) $\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{d}^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{d}^{-x}\right)$

b) $\cos x = \sin(\pi/2 + x)$

c) $x^a = \mathrm{e}^{a \ln x}$ for $a \in \mathbb{R}$
What does this imply for the derivative of $f(x) = x^{-1}$?

d) Use the result from (c) to proof the quotient rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{\big(g(x)\big)^2}$$

e) $\tan x = \dfrac{\sin x}{\cos x}$ and $\tanh x = \dfrac{\sinh x}{\cosh x}$

Find the derivative of $\ln x$ solely based on $\dfrac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^x = \mathrm{e}^x$.
Hint: Use that $x = \mathrm{e}^{\ln x}$ and take the derivative of both sides.

Problem 3.2. **Integrals of Elementary Functions**

In a moment we will also perform integrals to determine the work performed on a body when it is moving subject to a force. Practice you skills by evaluating the following integrals.

a) $\displaystyle\int_{-1}^{1} \mathrm{d}x\, (a+x)^2$

b) $\displaystyle\int_{-5}^{5} \mathrm{d}q\, (a + b\, q^3)$

$\displaystyle\int_{0}^{B} \mathrm{d}k\, \tanh^2(kx)$

c) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x/L}$

d) $\displaystyle\int_{-L}^{L} \mathrm{d}y\, \mathrm{e}^{-y/\xi}$

e) $\displaystyle\int_{0}^{L} \mathrm{d}z\, \frac{z}{a + b\, z^2}$

f) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\, x\, \mathrm{e}^{-x^2/(2Dt)}$

g) $\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}\ell\, \ell\, \mathrm{e}^{-\ell^2/(2Dt)}$

$\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}z\, x\, \mathrm{e}^{-z\, x^2}$

Except for the integration variable all quantities are considered to be constant.

Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

3.3 *Newton's Axioms*

In Section 4.1 we referred to a train compartment to point out that physical observations will be the same — irrespective of the velocity of its motion, as long as it is constant. A setting where we perform an experiment is denoted as reference frame, and reference frames that move with constant velocityare called inertial systems.

Definition 3.2: Reference Frames and Inertial Systems

A *reference frame* $(\boldsymbol{Q}, \{\hat{\boldsymbol{e}}_i(t), i = 1 \cdots D\})$ is an agreement about the, in general time dependent, position of the origin $\boldsymbol{Q}(t)$ of the coordinate system and a set of orthonormal basis vectors $\{\hat{\boldsymbol{e}}_i(t), i = 1 \cdots D\}$, that are adopted to indicate the positions of particles in a physical model.
The reference frame refers to an *inertial system* when it does not rotate and when it moves with a constant velocity, i. e. if and only if $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$ and $\dot{\hat{\boldsymbol{e}}}_i = \boldsymbol{0}$ for all $i \in \{1 \cdots D\}$.

Remark 3.1. A system at rest is an inertial frame that moves with the constant velocity $\boldsymbol{0}$.

Remark 3.2. Let $\boldsymbol{q} = (q_1, \dots q_D)$ be the coordinates of a particles, as specified in in the inertial frame $(\boldsymbol{Q}, \{\hat{\boldsymbol{e}}_i\})$, and $\boldsymbol{x} = (x_1, \dots x_D)$ its position given in the inertial frame $(\boldsymbol{X}, \{\hat{\boldsymbol{n}}_i\})$. Then

$$\boldsymbol{q} = \boldsymbol{Q} + \sum_{i=1}^{D} q_i\, \hat{\boldsymbol{e}}_i = \boldsymbol{X} + \sum_{i=1}^{D} x_i\, \hat{\boldsymbol{n}}_i\,.$$

3.3.1 1st Law

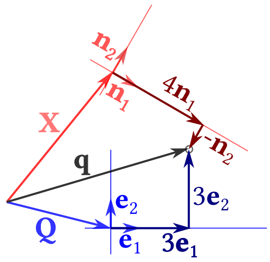


Figure 3.1: Graphical illustration of the description of a position from the perspective of two different reference frames, $\boldsymbol{q} = \boldsymbol{Q} + 3\,\hat{\boldsymbol{e}}_1 + 3\,\hat{\boldsymbol{e}}_2 = \boldsymbol{X} + 4\,\hat{\boldsymbol{n}}_1 - \hat{\boldsymbol{n}}_2$ with the notations of Remark 3.2.

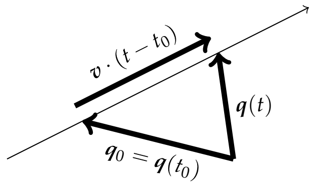
As long as a reference frame moves with a constant velocity, it feels like at rest. Physical measurements can only detect acceleration. This is expressed by

Axiom 3.1: Newton's 1st law

The velocity of a particle moving in an inertial system is constant, unless a (net) force is acting on the particle,

$$\begin{aligned} \forall t \geq t_0 : \boldsymbol{F}(t) = \boldsymbol{0} \quad &\Leftrightarrow \quad \dot{\boldsymbol{q}}(t) = \boldsymbol{v} = \text{const} \\ &\Leftrightarrow \quad \boldsymbol{q}(t) = \boldsymbol{q}_0 + \boldsymbol{v}\,(t - t_0) \end{aligned}$$

as sketched in the margin.



The particle moves then in a straight line with a constant speed. Indeed, when a particle moves with the constant velocity $\boldsymbol{v} = \dot{\boldsymbol{q}}(t)$ in the reference frame $(\boldsymbol{Q}_1, \{\hat{\boldsymbol{e}}_i(t), i = 1 \cdots D\}$ then it is at rest in the alternative reference frame $(\boldsymbol{Q}_2, \{\hat{\boldsymbol{e}}_i(t), i = 1 \cdots D\}$ where $\boldsymbol{Q}_2 = \boldsymbol{Q}_1 + \boldsymbol{v}\,t$. Therefore, in the latter coordinate system the particle is at rest, and it will remain at rest when it is not perturbed by a net external force. After all,

$$\boldsymbol{q} = \boldsymbol{Q}_1 + \boldsymbol{v}\,t = \boldsymbol{Q}_2 + \boldsymbol{0}\,.$$

3.3.2 2nd Law

Newton's second law spells out how the velocity of the particle changes when there is a force.

Axiom 3.2: Newton's 2nd law

The change, $\ddot{\boldsymbol{q}}(t)$, of the velocity of a particle, $\dot{\boldsymbol{q}}(t)$, at position, $\boldsymbol{q}(t)$, is proportional to the sum of acting forces $\boldsymbol{F}_\alpha$ with a proportionality factor m,

$$m\,\ddot{\boldsymbol{q}}(t) = \sum_\alpha \boldsymbol{F}_\alpha(t)\,.$$

Remark 3.3. In general the time dependence of the forces can be decomposed into three contributions

a) An implicit time dependence, $\boldsymbol{F}(\boldsymbol{q}(t))$, when the force depends on the position, $\boldsymbol{q}(t)$ of the particle. For instance, for a Hookian spring with spring constant k one has, $\boldsymbol{F}(\boldsymbol{q}) = -k\,\boldsymbol{q}$

b) An implicit time dependence, $\boldsymbol{F}(\dot{\boldsymbol{q}}(t))$, when the force depends on the velocity, $\dot{\boldsymbol{q}}(t)$ of the particle. For instance, the sliding friction for a particle with mass m and friction coefficient γ is, $\boldsymbol{F}(\dot{\boldsymbol{q}}) = -m\,\gamma\,\dot{\boldsymbol{q}}$

c) An explicit time dependence when the force is changing in time. For instance, when pushing a child sitting on a swing one will only push when the swing is moving in forward direction.

Typically, one explicitly sorts out these dependencies and writes

$$\ddot{\boldsymbol{q}}(t) = m \sum_{\alpha} \boldsymbol{F}_\alpha(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)$$

Example 3.1: Particle moving in the gravitational field

The gravitational field induces a constant force $m\,\boldsymbol{g}$ on a particle with mass m. Let it have velocity $\boldsymbol{v}_0$ at time t_0 when it is taking off from the position $\boldsymbol{q}_0$. Then Newton's 2nd law states that $\ddot{q}(t) = \boldsymbol{g}$, and this equation must be solved subject to the initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}(t_0) = \boldsymbol{v}$. By working out the derivatives one readily checks that this is given for

$$\boldsymbol{q}(t) = \boldsymbol{q}_0 + \boldsymbol{v}\,(t - t_0) + \frac{1}{2}\,\boldsymbol{g}\,(t - t_0)^2$$

Example 3.2: Particle moving in a circle

Let a particle of mass m move with constant speed in a circle of radius R such that its position can be written as

$$\boldsymbol{q}(t) = \begin{pmatrix} R\ \cos(\omega t) \\ R\ \sin(\omega t) \end{pmatrix}$$

with a constant angular velocity ω. Then its velocity and acceleration take the form

$$\dot{\boldsymbol{q}}(t) = \begin{pmatrix} -\omega\, R\ \sin(\omega t) \\ \omega\, R\ \cos(\omega t) \end{pmatrix}$$

$$\text{and}\quad \ddot{\boldsymbol{q}}(t) = \begin{pmatrix} -\omega^2\, R\ \cos(\omega t) \\ -\omega^2\, R\ \sin(\omega t) \end{pmatrix} = -\omega^2\, R\, \boldsymbol{q}(t)$$

The speed is constant, taking the value $\sqrt{\dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}} = \omega\, R$.
The force is antiparallel to $\boldsymbol{q}$ with magnitude $m\,\omega^2\,R$.
Moreover, $\dot{\boldsymbol{q}} \cdot \boldsymbol{F} = 0$ at all times. Hence, the force only changes the direction of motion, and not the speed.

3.3.3 3rd Law

Newton's third law states that the reference frame does not matter for the description of the evolution of two particles, even when they interact with each other — i.e. when they exert forces on each other. Consider for instance the motion of two particles of the same mass m that reside at the positions $\boldsymbol{q}_1(t)$ and $\boldsymbol{q}_2(t)$. We decide to observe them from a position right in the middle between the two particles $\boldsymbol{Q} = \big(\boldsymbol{q}_1(t) + \boldsymbol{q}_2(t)\big)/2$. In the absence of external forces this is an inertial frame, such that $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$ according to Newton's first law. However, Newton's second law implies that also

$$\mathbf{0} = 2m\ddot{Q} = m\,\ddot{\boldsymbol{q}}_1 + m\,\ddot{\boldsymbol{q}}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2$$

where $\boldsymbol{F}_1 = m\,\ddot{\boldsymbol{q}}_1$ and $\boldsymbol{F}_2 = m\,\ddot{\boldsymbol{q}}_2$ are the forces acting on particle 1 and 2, respectively. Up to a change of sign the forces are the same, $\boldsymbol{F}_1 = -\boldsymbol{F}_2$. This action-reaction principle is stipulated by

Axiom 3.3: Newton's 3rd law

Forces act in pairs:

actio when a body A is pushing a body B with force $\boldsymbol{F}_{A\to B}$

reactio then B is pushing A with force $\boldsymbol{F}_{B\to A} = -\boldsymbol{F}_{A\to B}$,

and these forces are always balanced, $\boldsymbol{F}_{A\to B} + \boldsymbol{F}_{B\to A} = \mathbf{0}$.

Example 3.3: Fixing a hammock at a tree

When you lie in a hammock that is fixed at a tree, your hammock exerts a force $\boldsymbol{F}_H$ on the tree (actio). The hammock stays where it is because the tree pulls back with exactly the same force $-\boldsymbol{F}_T$, up to a change of sign (reactio), and, in turn, this force can be written as the sum of two components accounting for the normal force F_N of the tree on the rope and a friction force F_f that prevents the rope from sliding down the tree.

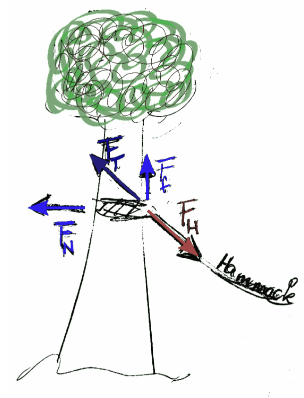


Figure 3.2: Graphical illustrations of forces involved in hanging a hammock on a tree, Example 3.3.

Example 3.4: Ice skaters

- When two ice skaters of the same mass push each other starting from a position at rest, then they will move in opposite directions with the same speed (unless they brake).
- When they have masses m_1 and m_2 their velocities will be related by $m_1\,\boldsymbol{v}_1 + m_2\,\boldsymbol{v}_2 = \mathbf{0}$ because $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \mathbf{0}$ initially, and $m_1\,\dot{\boldsymbol{v}}_1 + m_2\,\dot{\boldsymbol{v}}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \mathbf{0}$ at any instant of time. As long as they push, the velocities are non-zero and speed increases. When they slide there is no force any longer, and they go at constant speed—except for the impact of friction of the scates on the ice.

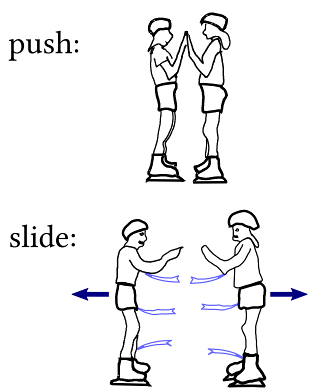


Figure 3.3: Graphical illustrations of motion of the two ice-skaters of Example 3.4.

Example 3.5: Water Rocket

A water rocket receives its thrust by the repulsive force in response of accelerating and releasing a water jet. Let M the mass of a rocket at a given time, and V_R its speed. To determine the acceleration of the rocket we consider a short time interval Δt where water of mass ΔM is ejected with speed v_f. In the absence of gravitation the momentum balance implies

$$
\begin{aligned}
M\,V_R &= (M - \Delta M)\,(V_R + \Delta V_R) + \Delta M\,(V_R - v_f) \\
\Leftrightarrow \qquad 0 &= M\,\Delta V_R - \Delta M\,v_f
\end{aligned}
$$

Now we observe that $\Delta M \;=\; a\,\rho, v_f\,\Delta t$ where a is the cross section of the ejected jet, and ρ the mass density of the ejected water:

$$
M\,\frac{\Delta V_R}{\Delta t} = a\,\rho, v_f^2 + a\,\rho, v_f\,\frac{\Delta V_R}{\Delta t}\,\Delta t
$$

and in the limit of small time increments $\Delta t \;\to\; 0$ we obtain the force F_R that is accelerating the rocket

$$
F_R = M\,\dot{V}_R = a\,\rho, v_f^2
$$

The rocket trajectory results from interplay of gravity and F_R. One case will be discussed as worked example at the end of this chapter, in Section 3.5. Solving the general case has been suggested as an instructive computer-based example for teaching mechanics (Gale, 1970; Finney, 2000). Instructions about how to build and discuss the rocket in school is available from the NASA and the instructables community.

Figure 3.4: Launching a water rocket, as introduced in Example 3.5.

3.3.4 *Punchline*

Newton's equations are stated nowadays in terms of derivatives, a concept in calculus that has been pioneered by Leibniz.[2] In this language they take the following form for a particle of mass m that is at position $\boldsymbol{q}(t)$ at time t,

$$
\begin{aligned}
\dot{\boldsymbol{q}}(t) &= \boldsymbol{v}(t) \\
\dot{\boldsymbol{v}}(t) &= \frac{1}{m}\,\boldsymbol{F}_{\text{tot}}(\boldsymbol{q}(t), \boldsymbol{v}(t), t)
\end{aligned}
$$

Prior to Newton, physical theories adopted the Aristotelian point of view that $\boldsymbol{v}$ is proportional to the force. Indeed in those days many scientists were regularly inspecting mines, and from the perspective of pushing mine-carts is is quite natural to assert that their velocity is proportional to the pushing force. Galileo's achievement is to add the 'tot' of the force side of the equation, pointing out that there also is a friction force acting on the mine-cart. Newton's achievement is to add the 'dot' on the left side of the equation, stating that the velocity stays constant when the pushing force and the friction force balance.

[2] Even thought the principles have been understood by Newton which lead to a very long fight for authorship and fame.

Example 3.6: Pushing a mine-cart

The motion of the mine-cart is one-dimensional along its track such that the position, q, velocity, x, and forces are one-dimensional, i. e. scalar functions. Once the mine-cart is moving it experiences a friction force $\boldsymbol{F}_f = -\gamma\, \boldsymbol{v}$, that (to a first approximation) is proportional to its velocity, $\boldsymbol{v}$. Now, let the mine-worker push with a constant force F_M such that

$$\ddot{q} = \dot{v} = F_{\text{tot}} = F_M - \gamma\, v\,.$$

The mine-cart travels with constant velocity $\dot{v} = 0$, when the attacking forces balance, i. e. for $v_c = F_M/\gamma$.
For a different initial velocity, $v(t_0) = v_0$, one finds an exponential approach to the asymptotic velocity,

$$v(t) = v_c + \big(v_0 - v_c\big)\ \mathrm{e}^{-\gamma\,(t-t_0)}$$

After all, $v(t_0) = v_c + \big(v_0 - v_c\big) = v_0$ and

$$\dot{v}(t) = \big(v_0 - v_c\big)\ (-\gamma)\, \mathrm{e}^{-\gamma\,(t-t_0)} = -\gamma\ \big(v(t) - v_c\big) = -\gamma\ v(t) + F_M$$

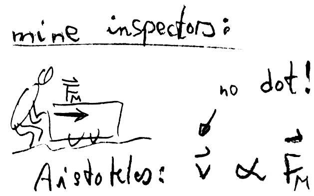


Figure 3.5: Illustration about the pre-Newtonian understanding of the relation between force and the velocity of a body.

The advantage of the Newtonian approach above earlier modeling attempts is that is makes a quantitative prediction about the asymptotic velocity, and that it also addresses the regime where the velocity is changing, e. g. when the mine-cart is taking up speed.

3.3.5 *Self Test*

Problem 3.3. **Terminal velocity for turbulent drag**

Rather than a friction of the type of the mine-cart, a golf ball experiences a drag force

$$\boldsymbol{F}_d = -\frac{\rho |u|^2}{2} c_d A\, \hat{\boldsymbol{u}}$$

where A is the cross section of the ball, ρ the density of air, $\boldsymbol{u}$ the velocity of the golf ball, and $c_d \simeq 0.5$ the drag coefficient.

a) The drag coefficient is a dimensionless number that depends on the shape of the object that experiences drag. For the rest the expression for the drag force follows from dimensional analysis. Verify this claim.

b) A slightly more informed derivation of $\boldsymbol{F}_d$ introduces also the diameter D of the golf ball and states that drag arises because the ball has to push air out of its way. When moving it has to push air out of the way at a rate $A\, u$. The air was at rest initially and must move roughly with a velocity u to get out of the way. Subsequently, its kinetic energy is lost. Check out, how this leads to the expression provided for $\boldsymbol{F}_d$.

c) What is the terminal velocity of a golf ball that is falling out of the pocket of a careless hang glider?

d) Use dimensional analysis to estimate the distance after which the ball acquires its terminal velocity, and how long it takes to reach the velocity.

Problem 3.4. **Orbit of the Moon around Earth**

The Moon is circling around Earth due to the gravitational force of modulus

$$F_{ME} = \frac{G M_E M_M}{R_{ME}^2}$$

where $G = \frac{2}{3} \times 10^{-12} \mathrm{m}^3/\mathrm{kg\,s}$ is the gravitational constant, $M_E \simeq 6 \times 10^{24}$ kg and $M_M \simeq \frac{3}{4} \times 10^{-23}$kg are the masses of Moon and Earth, respectively, and $R_{ME} = \frac{7}{4} \times 10^6$m is the distance from Earth to Moon.

a) Calculate the force that Moon is experiencing due to the Earth. Compare it to the gravitational acceleration $g \simeq 10\,\mathrm{m/s^2}$ scaled by $(R_{ME}/R_E)^2$ where $R_E = 2\pi \times 10^6$m is the Earth radius. What is the rational of the scaling by this factor?

b) Assume that the Moon trajectory is circular and identify F_{ME} with the centripetal force that keeps the moon on its orbit. What does this tell about the dependence of the period T of the motion on G, R_{ME} and the masses.

c) Evaluate T and compare it to the duration of a month.

Problem 3.5. **Pulling a Cow.**

A child is pulling a toy cow with a force of $F = 5\,\mathrm{N}$. The cow has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.

[2] For this angle one has $\tan\theta \approx 3/4$.

Children's Museum of Indianapolis, CC BY-SA 3.0

a) Describe the motion of the cow when there is no friction. In the beginning the cow is at rest.

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the cow.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

3.4 *Constants of motion (CM)*

In the previous section we saw that Newton's laws can be expressed as equations relating the second derivative of the position of a particle to the forces acting on the particle. The forces are determined as part of setting up the physical model. Subsequently, determining the time dependence of the position is a mathematical problem. Often it can be solved by finding constraints on the solution that must hold for all times. Such a constraint is called a

Definition 3.3: Constant of motion

A function $\mathcal{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is a *constant of motion (CM)* iff its time derivative vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0$$

It provides us with an opportunity to take a closer look at the expressions that emerge when taking derivatives of functions with arguments that are vectors. In order to evaluate the time derivative of $\mathcal{C}$ we write $\boldsymbol{q} = (q_1, \ldots\ldots, q_D)$, and apply the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(q_1(t), \cdots, q_D(t), \dot{q}_1(t), \cdots, \dot{q}_D(t), t) \\ &= \sum_{i=1}^{D} \frac{\mathrm{d}q_i}{\mathrm{d}t}\,\frac{\partial \mathcal{C}}{\partial q_i} + \sum_{i=1}^{D} \frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\,\frac{\partial \mathcal{C}}{\partial \dot{q}_i} + \frac{\partial \mathcal{C}}{\partial t} \qquad (3.4.1)\end{aligned}$$

In this expression the operation ∂ is called *'partial'*, and the derivative $\partial\mathcal{C}/\partial q_i$ is denoted as partial derivative of $\mathcal{C}$ with respect to q_i. For the purpose of calculating the partial derivative, we consider $\mathcal{C}$ to be a function of only the single argument q_i. For sake of a more compact notation we also write $\partial_{q_i}\mathcal{C}$ rather than $\partial\mathcal{C}/\partial q_i$. Moreover, when it is not clear from the context which conditions are adopted, they can explicitly be stated as subscript of a vertical bar to the right of the derivative (or even square brackets).

Example 3.7: Partial derivatives

For $f(x, y) = x/\sqrt{x^2 + y^2}$ and $R = \sqrt{x^2 + y^2}$ we have

$$\begin{aligned}\partial_x f(x, y)\big|_y &= \frac{1}{\sqrt{x^2 + y^2}} - \frac{x^2}{\left(x^2 + y^2\right)^{3/2}} = \frac{y^2}{R^3} \\ \partial_x f(x, y)\big|_R &= \partial_x \left[\frac{x}{R}\right]_R = \frac{1}{R}\end{aligned}$$

A compact notation that allows us to state the expression of Equation (3.4.1) in a more transparent way is achieved as follows: We observe that the expressions in the sums amount to writing out in components a scalar product of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ with vectors that are obtained by the partial derivatives. These vectors are denoted *gradients* with respect to $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, and they will be written as

$$\nabla_{\boldsymbol{q}}\,\mathcal{C} = \begin{pmatrix}\partial_{q_1}\mathcal{C} \\ \vdots \\ \partial_{q_D}\mathcal{C}\end{pmatrix} \qquad \text{and} \qquad \nabla_{\dot{\boldsymbol{q}}}\,\mathcal{C} = \begin{pmatrix}\partial_{\dot{q}_1}\mathcal{C} \\ \vdots \\ \partial_{\dot{q}_D}\mathcal{C}\end{pmatrix}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \dot{\boldsymbol{q}} \cdot \nabla_{\boldsymbol{q}}\,\mathcal{C} + \ddot{\boldsymbol{q}} \cdot \nabla_{\dot{\boldsymbol{q}}}\,\mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

In terms of the phase-space coordinates $\mathbf{\Gamma} = (\boldsymbol{q}, \dot{\boldsymbol{q}})$ one can also adopt the even more compact notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \dot{\mathbf{\Gamma}} \cdot \nabla_{\Gamma}\, \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

We make use of these derivatives while introducing some important physical quantities that are constants of the motion in specific settings.

3.4.1 *The kinetic energy*

When no forces are acting on a particle, $\boldsymbol{F}_{\text{tot}} = \mathbf{0}$, it moves with constant velocity. All functions that depend only on the velocity will then be constant. In particular this holds for the kinetic energy, T, that will play a very important role in the following.

Theorem 3.2: Conservation of kinetic energy

The *kinetic energy* $T = \dfrac{m}{2}\, \dot{\boldsymbol{q}}^2$ of a particle is conserved iff no net force acts on the particle, i. e. iff $\boldsymbol{F}_{\text{tot}} = \mathbf{0}$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}T &= \frac{m}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\sum_i \dot{q}_i \cdot \dot{q}_i = m\,\sum_i \dot{q}_i \cdot \ddot{q}_i \\ &= m\dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \cdot (m\,\ddot{\boldsymbol{q}}) = \dot{\boldsymbol{q}} \cdot \boldsymbol{F}_{\text{tot}} = 0\end{aligned}$$

In the last two steps we used Newton's 2nd law, and the assumption that $\boldsymbol{F}_{\text{tot}} = \mathbf{0}$. □

3.4.2 *Work and total energy*

From a physics perspective, work is performed when a body is moved in the presence of an external force.

- When the force $\boldsymbol{F}$ is constant along a path of displacement $\boldsymbol{s} = \boldsymbol{q}_1 - \boldsymbol{q}_0$, from a position $\boldsymbol{q}_0$ to the position $\boldsymbol{q}_1$, then the work W amounts to the scalar product $W = \boldsymbol{F} \cdot \boldsymbol{s}$.

- When the force changes upon moving along the path, we parameterize the motion along the path by time, $\boldsymbol{q}(t)$, with $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\boldsymbol{q}(t_1) = \boldsymbol{q}_1$ and break it into sufficiently small pieces $\boldsymbol{s}_i = \dot{\boldsymbol{q}}(t_i)\,\Delta t$ where the force $\boldsymbol{F}_i = \boldsymbol{F}(t_i)$ may be assumed to be constant. Then

  $$W = \sum_i \boldsymbol{F}_i \cdot \boldsymbol{s}_i = \lim_{\Delta t \to 0} \boldsymbol{F}_i \cdot \dot{\boldsymbol{q}}\,\Delta t = \int_{t_0}^{t_1} \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int_{\boldsymbol{q}(t)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$$

  The last equality should be understood here as a definition of the final expression that is interpreted here in the spirit of the substitution rule of integration.

Definition 3.4: Work and Line Integrals

The *work*, W, of a particle that performs a path $\boldsymbol{q}$ under the influence of a force $\boldsymbol{F}(t)$ amounts to the result of the *line integral*

$$W = \int_{\boldsymbol{q}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$$

When the path is parameterized by time, then W amounts to the time integral of dissipated power $P(t) = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)$,

$$W = \int \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t) \; \mathrm{d}t = \int P(t) \; \mathrm{d}t$$

Remark 3.4. The scalar product $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$ or $P(t) = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)$ singles out only the action of the force parallel to the trajectory. The perpendicular component does not perform work. Hence, a force that is always acting perpendicular to the velocity, i. e. perpendicular to the path of the particle, does not perform any work,

$$W = \int \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t) \; \mathrm{d}t = \int 0 \; \mathrm{d}t = 0$$

It only changes the direction of motion.

The calculation of work simplifies dramatically when the force can be written as gradient of another function, Φ.

Definition 3.5: Potentials and Conservative Forces

A force $\boldsymbol{F}(\boldsymbol{q})$ that can be expressed as the negative gradient of a function $\Phi(\boldsymbol{q})$,

$$\boldsymbol{F}(\boldsymbol{q}) = -\nabla\Phi(\boldsymbol{q}) = -\begin{pmatrix} \partial_{q_1}\Phi(q_1, \ldots, q_D) \\ \vdots \\ \partial_{q_D}\Phi(q_1, \ldots, q_D) \end{pmatrix}$$

is called a *conservative force* and the function Φ is the *potential* associated to the force.

Remark 3.5. Conservative forces only depend on position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{q})$. They neither explicitly depend on time nor on the velocity $\boldsymbol{q}$.

Theorem 3.3: Work for conservative forces

For conservative forces, $\boldsymbol{F} = -\nabla\Phi(\boldsymbol{q})$, the work for a path $\boldsymbol{q}(t)$ from $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$ amounts to the difference of the potential evaluated at the initial and at the final point of the path

$$W = \int_{\boldsymbol{q}(t)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q} = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1)$$

Proof.

$$\begin{aligned} W &= \int_{t_0}^{t_1} \boldsymbol{F} \cdot \dot{\boldsymbol{q}} \; \mathrm{d}t = -\int_{t_0}^{t_1} \nabla\Phi \cdot \; \dot{\boldsymbol{q}} \; \mathrm{d}t \\ &= -\int_{t_0}^{t_1} \sum_i \frac{\partial \Phi}{\partial q_i} \, \frac{\partial q_i}{\partial t} \; \mathrm{d}t = -\int_{t_0}^{t_1} \frac{\mathrm{d} \; \Phi}{\mathrm{d}t} \; \mathrm{d}t \\ &= -\big(\Phi(\boldsymbol{q}(t_1)) - \Phi(\boldsymbol{q}(t_0))\big) = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1) \end{aligned}$$

□

Remark 3.6. The work performed along a closed path vanishes for conservative forces. After all, in that case $\boldsymbol{q}_1 = \boldsymbol{q}_0$ such that $W = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1) = 0$.

Example 3.8: Gravitational Potential

For a particle of mass m gravity on the Earth surface gives rise to a force of magnitude $\boldsymbol{F}(x, y, z) = -m\, g\, \hat{\boldsymbol{z}}$ that can be derived from the potential $\boldsymbol{\Phi}(x, y, z) = m\, g\, z$,

$$-\nabla\Phi_1(x, y, z) = \begin{pmatrix} -\partial_x \Phi(x, y, z) \\ -\partial_y \Phi(x, y, z) \\ -\partial_z \Phi(x, y, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -m\, g \end{pmatrix} = \boldsymbol{F}(x, y, z)$$

Far away, at a position $\boldsymbol{q} = (q_1, q_2, q_3)$ from the center of Earth, gravity induces a force $\boldsymbol{F}(\boldsymbol{q}) = -G\, M_E\, m\, \boldsymbol{q}/|\boldsymbol{q}|^3$ on a body of mass m. This force can be obtained as

$$-\nabla\phi_2(\boldsymbol{q}) = \nabla \frac{G\, M_E\, m}{\sqrt{q_1^2 + q_2^2 + q_3^2}} = G\, M_E\, m \begin{pmatrix} \partial_{q_1} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \\ \partial_{q_2} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \\ \partial_{q_3} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \end{pmatrix}$$

$$= G\, M_E\, m \begin{pmatrix} \frac{-q_1}{[q_1^2+q_2^2+q_3^2]^{-3/2}} \\ \frac{-q_2}{[q_1^2+q_2^2+q_3^2]^{-3/2}} \\ \frac{-q_3}{[q_1^2+q_2^2+q_3^2]^{-3/2}} \end{pmatrix} = \frac{-G\, M_E\, m}{[q_1^2 + q_2^2 + q_3^2]^{3/2}}\, \boldsymbol{q} = \boldsymbol{F}(\boldsymbol{q})$$

Remark 3.7. According to Theorem 3.3 differences of the value of the potential, and hence also the functional dependence of the potential up to some constant, are related to the work performed in the potential. Hence, the potential is an observable that must not depend on the choice of the coordinate system. Therefore it can always be expressed in terms of scalar products. For the potentials in Example 3.8 this is achieved by writing

$$\Phi_1(\boldsymbol{q}) = m\, \boldsymbol{g} \cdot \boldsymbol{q} \quad \text{with} \quad \boldsymbol{g} = (0, 0, g)$$
$$\Phi_2(\boldsymbol{q}) = -G\, M_E\, m / \sqrt{\boldsymbol{q} \cdot \boldsymbol{q}}$$

Remark 3.8. One can make use of the properties of scalar products to reduce the computational work to determine the force for a given potential by working out the component i of the gradient where i is can be any index of the vector. For conciseness we also write then ∂_i for the partial derivative with respect to component q_i of the argument $\boldsymbol{q}$ of $\Phi(\boldsymbol{q})$.

For the potentials in Example 3.8 this works as follows

$$-\partial_i \Phi_1(\boldsymbol{q}) = -m\, \partial_i \sum_j g_j q_j = -m \sum_j g_j \delta_{ij} = -m g_i$$
$$-\partial_i \Phi_2(\boldsymbol{q}) = G\, M_E\, m\, \partial_i \Big[\sum_j q_j^2\Big]^{-1/2} = \frac{-G\, M_E\, m\, q_i}{\big[\sum_j q_j^2\big]^{3/2}}$$

In particular in the second case the advantage is evident.

Example 3.9: Falling men and cat

When a cat, that has a mass of $m = 3\,\mathrm{kg}$, falls from a balcony in the fourth floor, i. e. from a height $H \simeq 4 \times 3\,\mathrm{m} = 12\,\mathrm{m}$, the initial potential energy

$$V_{\mathrm{cat}} = mgH = 3\,\mathrm{kg} \times 10\,\mathrm{m/s^2} \times 12\,\mathrm{m} = 360\,\mathrm{kg\,m^2/s^2}$$

will be transformed into kinetic energy and then dissipated when the cat hits the ground.
To get an idea about this energy we compare it to the energy dissipated when a man of mass $M = 80\,\mathrm{kg}$, falls out of his bed that has a height of $h = 50\,\mathrm{cm}$,

$$V_{\mathrm{man}} = Mgh = 80\,\mathrm{kg} \times 10\,\mathrm{m/s^2} \times 0.5\,\mathrm{m} = 400\,\mathrm{kg\,m^2/s^2}$$

From the point of view of the dissipated energy the fall of the cat is not as bad as it looks at first sight.

Conservative forces are called conservative forces because motion in such a potential conserves the sum of the potential energy and the kinetic energy.

Theorem 3.4: Conservation of the total energy

The *total energy* $E = T + \Phi$ of a particle is conserved if it moves in a conservative force field $\boldsymbol{F} = -\nabla\Phi$.

Proof.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}\Phi}{\mathrm{d}t} = m\,\dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} + \nabla\Phi \cdot \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \cdot \underbrace{\left(m\ddot{\boldsymbol{q}} - \boldsymbol{F}\right)}_{=\,\mathbf{0}} = \mathbf{0}$$

In the third equality we used that the force is conservative, and in the final step, we used Newton's second law which states that $m\ddot{\boldsymbol{q}} = \boldsymbol{F}$. □

Example 3.10: Accidents at work and on the street

A paramedic emergency ambulance receives two calls from an accident site:
i. a craftsmen fell from a roof of height H
ii. a teenager hit a tree with his motorcycle with a speed v
For which height does the energy of the craftsman approximately match the one of the motor cyclist when he drove
in the city, $v_C = 50\,\mathrm{km/h}$,
outside the city, $v_L = 100\,\mathrm{km/h}$,
on a German autobahn with $v_A = 150\,\mathrm{km/h}$
or was really speeding with $v_S = 200\,\mathrm{km/h}$.
We assume that they both have comparable mass.
Energy conservation entails that we have to compare the potential energy V_{worker} of the craftsman on the roof and the

kinetic energy of the teenager on the motorcycle T_{teenager},

$$mgH = V_{\text{worker}} = T_{\text{teenager}} = \frac{m}{2}\, v^2 \quad \Leftrightarrow \quad H = \frac{v^2}{2g}$$

Hence we find

v	50 km/h	100 km/h	150 km/h	200 km/h
H	12 m	50 m	110 m	200 m
floor	4	16	36	64

Most likely, the teenager will encounter more severe injuries, unless the craftsman is working on a really high building.

3.4.3 *Momentum*

Theorem 3.5: Conservation of momentum

The *momentum* $\boldsymbol{P} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$ is conserved if no net force $\boldsymbol{F}_{\text{tot}}$ acts on the system.

Proof.
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = \sum_{i=1}^{N} m_i \ddot{\boldsymbol{q}}_i(t) = \sum_{i<j} (\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji}) + \sum_i \boldsymbol{F}_i = \boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$$

where $\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji}$ vanishes due to Newton's third law, and the net external force is zero by assumption. □

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Figure 3.6: Newton's cradle. When the excited ball to the right is released it will come down, hit the rightmost ball that is hanging down at rest, the momentum is transferred to the leftmost ball, and that is moving up to (almost) as much to the left as the initial ball was excited to the right. Its motion reverses, and the motion repeats.

Example 3.11: One-dimensional collisions

We consider two steel balls that can freely move along a line. They have masses m_1 and m_2 and reside at positions x_1 and x_2, respectively. Initially ball two is at rest in the origin, and ball one is approaching from the right with a constant speed v_1. What is the speed of the balls after the collision? Before and after the collision the particles feel no forces such that their velocity is constant. We assume that the collision is elastic such that energy is preserved. Hence,

$$\begin{aligned} \text{before collision} &= \text{after collision} \\ m_1\, v_1 &= m_1\, v_1' + m_2\, v_2' \\ \frac{m_1}{2}\, v_1^2 &= \frac{m_1}{2}\, (v_1')^2 + \frac{m_2}{2}\, (v_2')^2 \end{aligned}$$

where the prime indicates the post-collision velocities. From the momentum balance we obtain $m_2\, v_2' = m_1\, (v_1 - v_1')$. When this is used to eliminate v_2' from the energy balance, a straightforward calculation provides

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}\, v_1 \qquad \text{and} \qquad v_2' = \frac{2\, m_1}{m_1 + m_2}\, v_1$$

In particular, when the two particles have the same mass one obtains that $v'_1 = 0$ and $v'_2 = v_1$ which is beautifully exemplified by the dynamics of Newton's cradle.

3.4.4 Angular Momentum

In the immediate vicinity of the collisions the balls in Newton's cradle perform a motion along a horizontal line, as discussed in Example 3.11. However, during the excursions to the left and right they follow a circular track where the chains act as arms and their suspension as fulcrum of the circular motion. In such settings it is often desirable to also consider the evolution of the angular momentum.

Theorem 3.6: Conservation of angular momentum

The *angular momentum* $\boldsymbol{L} = \sum_{i=1}^{N} m_i \boldsymbol{q}_i(t) \times \dot{\boldsymbol{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$ is conserved if no external forces act on the system and if the interaction forces between pairs of particles act parallel to the line connecting the particles.

Proof.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} &= \sum_{i=1}^{N} m_i \Big(\dot{\boldsymbol{q}}_i(t) \times \dot{\boldsymbol{q}}_i(t) + \boldsymbol{q}_i(t) \times \ddot{\boldsymbol{q}}_i(t) \Big) \\
&= \sum_{i<j} \Big(\boldsymbol{q}_i(t) \times \boldsymbol{f}_{ij} + \boldsymbol{q}_j(t) \times \boldsymbol{f}_{ji} \Big) \\
&= \sum_{i<j} \Big(\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t) \Big) \times \boldsymbol{f}_{ij} = \boldsymbol{0}
\end{aligned}
$$

where we used that $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$ due to Newton's third law, and that $(\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t))$ is parallel to $\boldsymbol{f}_{ij}$ by assumption on the particle interactions. □

Example 3.12: Determine the speed of a bullet.

In a CSI lab one tests the speed of a bullet by shooting it into a rotor where a mass $M = 1\,\mathrm{kg}$ can move horizontally with minimal fraction on an arm with length $L = 1\,\mathrm{m}$. For a bullet of a mass $m = 8\,\mathrm{g}$ we find a rotation frequency $f = 0.16\,\mathrm{Hz}$. What is the muzzle velocity of the gun? During the collision the bullet gets stuck in the rotor mass. Before and after the collision the angular momentum thus is

$$
\begin{aligned}
& m\,R\,v = (m+M)\,R^2\,\omega = (m+M)\,R^2\,2\pi\,f \\
\Leftrightarrow \qquad & v = \frac{m+M}{m}\,2\pi\,f\,R = \frac{1008}{8} \times 2\pi \times 0.16\,\mathrm{m/s} \simeq 125\,\mathrm{m/s}
\end{aligned}
$$

3.4.5 Self Test

Problem 3.6. **Derivatives of common composite expressions**
Evaluate the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}(a+x)^b$

b) $\frac{\partial}{\partial x}(x+b\,y)^2$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x+y(x))^2$

d) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta(t)$

e) $\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin\theta(t)\ \cos\theta(t)\big)$

f) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\big(2\theta(t)\big)$

g) $\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{a+b\,z^2}$

h) $\frac{\partial}{\partial x_3}\left[\sum_{j=1}^{6} x_j^2\right]^{-1/2}$

i) $\frac{\partial}{\partial y_1}\ln(\boldsymbol{x}\cdot\boldsymbol{y})$

In these expressions a and b are real constants, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are 6-dimensional vectors.


Rex Hammock from USA/ wikimedia CC BY-SA 2.0

Problem 3.7. **Anvil shooting.**

Anvil shooting is a tradition in some US communities to celebrate St. Clement's Day, honoring Pope Clement I, the patron saint of blacksmiths and metalworkers. Typical anvils have a mass of about 150 kg and they are shot up to a height of 60 m. Which energy must the gun powder release to the anvil for such a feat?

Problem 3.8. **Running Mothers.**

In the Clara Zetkin Park one regularly encounters blessings[3] of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their youngs in pouches.

[3] Look up "terms of venery" if you ever run out of collective nouns.

a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.
The carriages suffer from friction. Let the friction coefficient be $\gamma = 0.3$.
When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.

b) How does the running speed matter in this discussion?

c) How does the mass of the babies/youngs make a difference?

Problem 3.9. **The sledgehammer experiment.**

In his magnificent book "Thinking Physics" Lewis Carroll Epstein (2009) sets out a class room experiment that he used to perform in his physics class: He placed an anvil on his chest and asked a student from the audience to hit the anvil with a sledge hammer as hard as he could manage. What will happen?

Epstein changed the way of presentation of this experiment when a very nervous student missed the anvil and hit his hand. Have a look into the book for the full story.

Problem 3.10. **The rotating chair experiment.**

The spin increases when an ice dancer pulls inwards arms and legs. This is illustrated in the picture of Yuko Kawaguti in the margin, and the physical principle has beautifully been demonstrated in a wikimedia movie by Oliver Zajkov from the Physics Institute at the University of Skopje.

a) Assume that a less careful experimenter starts his motion with a spin of 1 Hz, holding 5 kg barbells with stretched-out arms 1 m away from the rotation axis. Estimate his spin rate when he pulls in his arms till the barbells reach a distance of 20 cm from the rotation axis.

b) Which trajectory will they take when the careless experimenter gets dizzy and looses hold on the barbells?

deerstop, wikimedia, CC0

3.5 Worked example: Flight of an Earth-bound rocket

In order to illustrate the applications of Newton's laws we discuss now the flight of a rocket. We will deal with the case a) where the rocket is moving in vertical direction, b) where the fuel is ejected with a constant speed (or zero when it is exhausted), and c) where the rocket does not reach heights with a noticeable change of the gravitational acceleration. At the end of this section we discuss the impact of relaxing these assumption, and point to the literature for a further discussion.

Let V_R be the speed of the rocket. It is positive when the rocket goes up, and negative when it falls down. On the way down, its mass will be m. Initially, it has a mass $m + M_0$, where M_0 is mass of the fuel. As long as the rocket is firing, Newton's thirs law implies that

$$F_R = \big(m + M(t)\big)\, \dot{V}_R = a\,\rho\, v_f^2 - \big(m + M(t)\big)\, g$$

The first force on the right-hand side of this equation accounts for the recoil from ejection of the fuel (cf. Example 3.5) and the latter to gravitational acceleration. We also observed in Example 3.5 that the mass $M(t)$ of the remaining fuel at time t obeys the differential equation $\dot{M} = -a\,\rho\, v_f$ such that[4]

$$M(t) = M_0 - -a\,\rho\, v_f\, t\,.$$

Consequently, we find for the velocity of the rocket

$$\dot{V}_R(t) = -g + \frac{a\,\rho\, v_f^2}{m + M_0 - a\,\rho\, v_f\, t} \qquad \text{and} \qquad V_R(0) = 0$$

$$\Rightarrow \qquad V_R(t) = -g\, t - v_f\, \ln\left(1 - \frac{a\,\rho\, v_f\, t}{m + M_0}\right) \tag{3.5.1}$$

When all fuel is consumed, at time T, we have

$$0 = M_0 - a\,\rho\, v_f\, T \quad \Rightarrow \quad T = \frac{M_0}{a\,\rho\, v_f}\,,$$

and the rocket has aquired the speed

$$V_R(T) = -g\, T + v_f\, \ln\left(1 + \frac{M_0}{m}\right) = -g\, T + v_f\, \ln\frac{\mu}{1+\mu}\,.$$

[4] One easily checks that this expression is correct for the initial mass, $M(0) = M_0$ and its derivative agrees with $\dot{M}(t)$. The same applies also t the expressions for the speed and height of the rocket discussed below. In Chapter 4 we will discuss systematic approaches to find the solution. Problem 4.2 will give clues how the solutions are determined systematically.

In the last step we introduced the ratio $\mu = m/M_0$ that will turn out to be a useful abbreviation in forthcoming equations.

The rocket height $z(t)$ is obtained by observing that $\dot{z}(t) = V_R(t)$, which in turn is given by Equation (3.5.1). The solution where the rocket starts at height zero is the given by

$$\dot{z}(t) = V_R(t) \quad \text{and} \quad z(0) = 0$$
$$\Rightarrow \quad z(t) = -\frac{g\,t^2}{2} + v_f\,t + v_f\,T\left(\mu + 1 - \frac{t}{T}\right)\,\ln\left(1 - \frac{t/T}{1+\mu}\right)\,. \tag{3.5.2}$$

At time T this simplifies to

$$z(T) = -\frac{g\,T^2}{2} + v_f\,T\,\left[1 + \mu\,\ln\frac{\mu}{1+\mu}\right]$$

Starting from that position the rocket will perform a ballistic flight with initial velocity $V_R(T)$ that will add to its height another height increment of $V_R^2(T)/2g$. The additional height increment ΔH before the rocket reaches the crest of its height is found by energy conservation

$$m\,g\,\Delta H = \frac{m}{2}\,V_R^2(T) \quad \Rightarrow \quad \Delta H = \frac{V_R^2(T)}{2\,g}$$

Altogether the rocket therefore reaches the height H, given by

$$\begin{aligned} H &= \left[-\frac{g\,T^2}{2} + v_f\,T + \mu\,v_f\,T\,\ln\frac{\mu}{1+\mu}\right] \\ &\quad + \left[\frac{g\,T^2}{2} + v_f\,T\,\ln\frac{\mu}{1+\mu} + \frac{v_f^2}{2\,g}\,\ln^2\frac{\mu}{1+\mu}\right] \\ &= v_f\,T\,\left[1 + (1+\mu)\ln\frac{\mu}{1+\mu}\right] + \frac{v_f^2}{2\,g}\,\ln^2\frac{\mu}{1+\mu} \end{aligned} \tag{3.5.3}$$

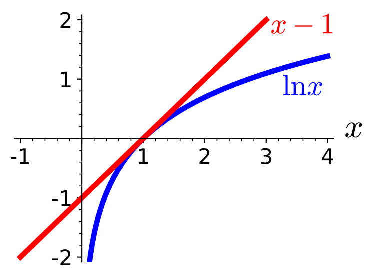


Figure 3.7: The function $x-1$ (red) is always larger than $\ln x$ (blue).

The expression in the square bracket is always negative, as one can see based on the inequality $\ln x \le x - 1$ shown Figure 3.7. Hence,

$$1 + (1+\mu)\ln\frac{\mu}{1+\mu} \le 1 + (1+\mu)\,\left(\frac{\mu}{1+\mu} - 1\right) = 0$$

The best strategy to achieve a large height is to go for a small T in order to suppress the first term in Equation (3.5.3) and large v_f to achieve large values of the second term.

When energy efficiency is a concern, e. g. when the rocket is used for a measurement of the atmosphere at height H, one might be interested to reach that height with minimum energy cost. This means one is interested to minimize the ratio of the potential energy of the rocket at height H and the the energy $M_0 v_f/2$ burned to deliver the freight,

$$\begin{aligned} \eta = \frac{m\,g\,H}{M_0 v_f^2/2} &= \frac{2\,m\,g\,T}{M_0\,v_f}\,\left[1 + (1+\mu)\ln\frac{\mu}{1+\mu}\right] + \frac{m}{M_0}\,\ln^2\frac{\mu}{1+\mu} \\ &= \frac{2\,g\,T}{v_f}\,\mu\,\left[1 + (1+\mu)\ln\frac{\mu}{1+\mu}\right] + \mu\,\ln^2\frac{\mu}{1+\mu} \end{aligned} \tag{3.5.4}$$

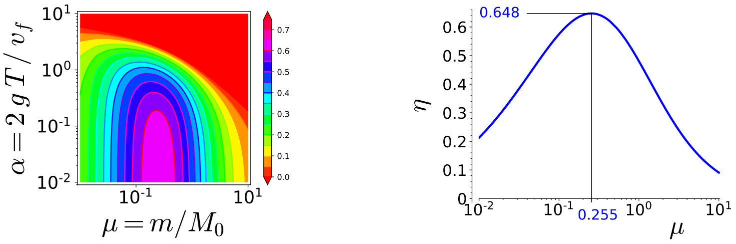


Figure 3.8: (left) Contour line for the efficiency, Equation (3.5.4), as function of τ and $m\,g\,T/v_f$. The maximum is taken for $m\,g\,T/v_f = 0$. (right) Plot of the μ dependence of the efficiency for $m\,g\,T/v_f = 0$. The maximum efficiency of $\eta_{\text{opt}} \simeq 0.648$ is obtained for $\mu_c \simeq 0.255$.

The efficiency is a function of μ and of the dimensionless number $\alpha = 2\,g\,T/v_f$. The countour lines of $\eta(\mu, \alpha)$ are plotted in the left panel of Figure 3.8.

Definition 3.6: Contour lines and iso-surfaces

The *contour lines* of a two variable function $f(x, y)$ are those lines in the (x, y)-plane, where $f(x, y)$ takes some constant value. More generally these lines are also called *isolines*, the two-dimensional surfaces where a three-variabel function $g(x, y, z)$ in the (x, y, z)-space takes constant values are called isosurfaces, and the $N - 1$-dimensional hypersurfaces of $\mathbb{R}^N$ where the function $h(\boldsymbol{q})$ with $\boldsymbol{q} \in \mathbb{R}^N$ takes a constant values will also be denoted as iso-surfaces.

The contour lines of the efficiency reveal that the maximum efficiency is optained for $\alpha = 0$, which can expected since the expression in square brackets was negative. The maximum efficience amounts therefore to the maximum of $\eta(\mu, \alpha = 0) = \mu \ \ln^2[\mu/(1+\mu)]$, which amounts to the root μ_c of the equation $2/(1+\mu) + \ln[\mu/(1+\mu)]$. Numerically it is found to be $\mu_c \simeq 0.255$. Hence, the maximum efficiency is obtained when the mass of the fuel M_0 is roughtly four times larger than the mass of the empty rocket. The maximum efficiency amounts then to

$$\eta_{\max} = \frac{4\,\mu_c}{(1+\mu_c)^2} \simeq 0.648\,.$$

Irrespective of the rocket design one can not transform more than $2/3$ of the energy of the fuel into potential energy of the rocket. The remaining energy is dissipated in the kinetic energy of the exhaust.

Further discussion of the trajectories of rockets can be found in Finney (2000); Gale (1970); Seifert *et al.* (1947). A discussion of water rockets that addresses the change of speed v_f of the ejected water was given in Kagan *et al.* (1995); Gommes (2010).

3.6 *Problems*

3.6.1 *Practicing Concepts*

Problem 3.11. **Car on an air-cushion**

We consider a car of mass $m = 20\,\mathrm{g}$ moving – to a very good approximation without friction – on an air-cushion track. There is a string attached to the car that moves over a roll and hangs vertically down on the side opposite to the car.

a) Sketch the setup and the relevant parameters.

b) Which acceleration is acting on the car when the string is vertically pulled down with a force of $F = 2\,\mathrm{N}$. Determine the velocity $v(t)$ and its position $x(t)$.

c) Determine the force acting on a 200 g chocolate bar, in order to get a feeling for the size of the force that was considered in (b).

d) Now we fix the chocolate bar at the other side of the string. The verlocity of the car can then be obtained based on energy conservation

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{m+M}{2}\, v^2 + Mgh = \text{konst},$$

where M is the mass of the chocolate bar. Is the acceleration the same of different as in the cases (b) and (c)? Provide an argument for your conclusion.

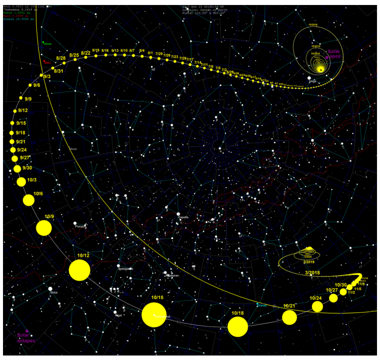


Figure 3.9: 'Oumuamua trajectory as seen by an observer on Earth.

Problem 3.12. **'Oumuamua.**

On 19 October 2017 astronomers at the Haleakala Observatory in Hawaii discovered 'Oumuamua, the first interstellar object observed in our solar system. It approached the solar system with a speed of about $v_I = 26\,\mathrm{km/s}$ and reached a maximum speed of $v_P = 87.71\,\mathrm{km/s}$ at its perihelion, i. e. upon closest approach to the sun on 9 September 2017.

a) Show that at the perihelion the speed and 'Oumuamua's smallest distance to the sun, D, obey the relation

$$\frac{v_P^2 - v_I^2}{2} = \frac{M_S\, G}{D}$$

while for the Earth we always have

$$\frac{4\pi^2 R}{T^2} \simeq \frac{M_S\, G}{R^2}$$

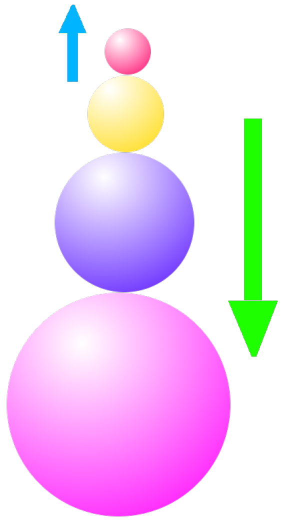
Here, M_S is the mass of Sun, R is the Earth-Sun distance, and $T = 1\,\text{year}$ is the period of Earth around Sun.

b) Show that this entails that $\dfrac{D}{R} = \dfrac{2\, v_E^2}{v_P^2 - v_I^2}$, where $v_E = 2\pi R/T$ is the speed of Earth around sun.

c) Use the relation obtained in (b) to determine D in astronomical units, and compare your estimate with the observed value $D = 0.25534(7)\,\mathrm{AU}$.

Problem 3.13. **Galilean cannon.**

In the margin we show a sketch of a Galilean cannon. Assume that the mass mass ratio of neighboring balls with always two, and that they perform elastic collisions.



The Children's Museum of Indianapolis, SteveBaker/wikimedia, CC BY-SA 3.0

a) Initially they are stacked exactly vertically such that their distance is negligible. Let the distance between the ground the lowermost ball be 1 m. How will the distance of the balls evolve prior to the collision of the lowermost ball with the ground?

b) After the collision with the ground the balls will move up again. Determine the maximum height that is reached by each of the balls.

Problem 3.14. **Motion in a harmonic central force field**

A particle of mass m and at position $\boldsymbol{r}(t)$ is moving under the influence of a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = -k\,\boldsymbol{r}\,.$$

a) We want to use the force to build a particle trap,[5] i. e. to make sure that the particle trajectories $\boldsymbol{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\boldsymbol{r}(t)| < B$ for all times t. What is the requirement on the sign of the constant k to achieve this aim?

[5] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

b) Determine the energy of the particle and show that its energy is conserved.

c) Demonstrate that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m\,\dot{\boldsymbol{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

Problem 3.15. **Collision with an elastic bumper.**

For consider two balls of radius R with masses m_1 and m_2 that are moving along a line. Their positions will be denoted as x_1 and x_2 in such a way that they touch when $x_1 = x_2$ and they do not feel each other when $x_1 < x_2$. When they run into each other, the balls can slightly be deformed such that the distance between their centers takes the value $2R - d$, and they experience a harmonic repulsive forces $\pm k\,d$. We will say then that $d = x_2 - x_1 < 0$.

a) Newton's equations for the two balls take the form

$$m_1\,\ddot{x}_1(t) = -k\,d(t) \qquad\qquad m_2\,\ddot{x}_2(t) = k\,d(t)$$

Show that this implies

$$\ddot{d} = -\omega^2 \, d$$

for some positive constant ω. How does ω depend on the spring constant k and on the masses m_1 and m_2?

b) Let $d(t) = -d_M \, \sin\big(\omega\,(t - t_0)\big)$ describe the deformation of the balls for a collision at $t = t_0$, and contact in the time interval $t_0 \leq t \leq t_R$. Verify that it is a solution of the equation of motion. At which time t_R will the particles release (i.e. there is no overlap any longer)? What is the maximum potential energy stored in the harmonic potential?

c) We consider initial conditions where particle 1 arrives with a constant velocity v_0 from the left, and particle 2 is at rest. What is the total kinetic energy in this situation? Assume that at most a fraction α of the kinetic energy is transferred to potential energy. What is the relation between v_0 and the maximum deformation d_M?

d) The velocity of the two particles at times $t_0 \leq t \leq t_R$ can now be obtained by solving the integrals

$$m_i \, \dot{x}_i(t) = m_i \, x_i(t_0) + (-1)^i \int_{t_0}^{t} \mathrm{d}t_I \, k \, d(t) \,, \quad \text{with } i \in \{1, 2\}$$

Why does this hold? Which values does $x_i(t_0)$ take? Solve the integral and show that

$$\dot{x}_1 = v_0 \, \Big[1 + \sqrt{\alpha\beta}\,\Big(\cos(\omega\,(t - t_0)) - 1\Big)\Big]$$
$$\dot{x}_2 = v_0 \, \frac{m_1}{m_2} \, \sqrt{\alpha\beta}\,\Big(\cos(\omega\,(t - t_0)) - 1\Big)$$

How does β depend on the masses?

e) Verify that release we have

$$\dot{x}_1 = v_0 \, \big(1 - 2\sqrt{\alpha\beta}\big)$$
$$\dot{x}_2 = v_0 \, \frac{2\,m_1}{m_2} \, \sqrt{\alpha\beta}\big)$$

Verify that these expressions comply to momentum conservation. Verify that the expressions obey energy conservation iff $\alpha = \beta = m_2/(m_1 + m_2)$.

f) What does this imply for particles of identical masses, $m_1 = m_2$? How does your result fit to the motion observed in Newton's cradle? What does it tell about the assumption of instantaneous collisions of balls that is frequently adopted in theoretical physics?

Problem 3.16. **Inelastic collisions, ballistics, and cinema heroes.**

Let us take a look at how cinema heroes shoot.

a) The title of Stanley Kubrick's movie *Full Metal Jacket* refers to full metal jacket bullets, i. e. projectiles as they were used in the M16 assault rifle used in the Vietnam war. Its bullets have a mass of 10 g and they set a 1 kg wooden block revolving at a 1 m arm into a 8 Hz motion. What is the velocity of the bullets?

The bullets of a 9 mm Luger pistol have a mass of 8 g and they are fired with a muzzle velocity of $350\,\mathrm{m\,s^{-1}}$. What is the resulting angular speed $\dot{\theta}$ of the wooden block?

b) Alternatively one can preform this measurement by shooting the bullet into a swing where a wooden block of mass M is attached to ropes of length ℓ. Initially it is at rest. Consider angular momentum conservation to determine its velocity immediately after impact. What does this tell about the kinetic energy immediately after the impact, and what about the maximum height of reached by the swing in its subsequent motion?

Let L be 2 m. Which mass is required to let the swing go up to the height of its spindle?
What does this tell about the recoil of the pistol and the rifle?
What do you think now about the Rambo shooting scene that you can find here on YouTube?

3.6.2 Mathematical Foundation

Problem 3.17. **Solving Integrals by Partial Integration**
Evaluate the following integrals by partial integration

$$\int \mathrm{d}x\, f(x)\, g'(x) = f(x)\, g(x) - \int \mathrm{d}x\, f'(x)\, g(x)$$

a) $\displaystyle\int_a^b \mathrm{d}x\, x\, \mathrm{e}^{kx}$ b) $\displaystyle\int_a^b \mathrm{d}x\, x^2\, \mathrm{e}^{kx}$ $\displaystyle\int_a^b \mathrm{d}x\, x^n\, \mathrm{e}^{kx}$, $n \in \mathbb{N}$

The integral c) can only be given as a sum over $j = 0, \ldots, n$.

Problem 3.18. **Substitution with Trigonometric and Hyperbolic Functions**
Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{q(x_1)}^{q(x_2)} \mathrm{d}q\, f(q) = \int_{x_1}^{x_2} \mathrm{d}x\, q'(x)\, f(q(x))$$

with a function $q(x)$ that is bijective on the integration interval $[x_1, x_2]$.

a) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}}$ by substituting $x = \sin\theta$

b) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}}$ by substituting $x = \sinh z$

c) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1+x^2}$ by substituting $x = \tan\theta$

d) $\displaystyle\int_a^b \mathrm{d}x \, \frac{1}{1-x^2}$ by substituting $x = \tanh z$

Problem 3.19. **Gradients and contour lines**

Determine the derivatives of the following functions.

a) Contour lines in the (x, y)-plane are lines $y(x)$ or $y(x)$ where a functions $f(x, y)$ takes a constant value. Sketch the contour lines of the functions

$$f_1(x, y) = (x^2 + y^2)^{-1} \qquad \text{and} \qquad f_2(x, y) = -x^2\, y^2$$

b) Determine the gradients $\nabla f_1(x, y)$ and $\nabla f_2(x, y)$.
Hint: The gradient $\nabla f_i(x, y)$ with $i \in \{1, 2\}$ is a vector $(\partial_x f_i(x, y), \partial_y f_i(x, y))$ that contains the two partial derivatives of the (scalar) function $f_i(x, y)$.

c) Indicate the direction and magnitude of the gradient by appropriate arrows in the sketch showing the contour lines. In which direction is the gradient pointing?

3.6.3 *Transfer and Bonus Problems, Riddles*

Problem 3.20. **Moeschenbroeks double-cone experiment.**

User:FA2010, Public domain

In the margin we show Moeschenbroeks double-cone experiment. The setup involves three angles:

1. The opening angle α between the two rails.
2. The angle ϕ of the rail surface with the horizontal.
3. The opening angle θ of the cone.

When it is release from the depicted position the cone might move to the right, to the left, and it could stay where it is. How does the selected direction of motion depend on the choice of the three angles?

Problem 3.21. **Coulomb potential and external electric forces.**

We consider the Hydrogen atom to be a classical system as suggested by the Sommerfeld model. Let the proton be at the center of the coordinate system and the electron at the position $\boldsymbol{r}$. The interaction between the proton and the electron is described by the Coulomb potential $\alpha/|\boldsymbol{r}|$. In addition to this interaction there is a constant electric force acting, that is described by the potential $\boldsymbol{F} \cdot \boldsymbol{r}$. Altogether the motion of the electron is therefore described by the potential

$$U = -\frac{\alpha}{|\boldsymbol{r}|} - \boldsymbol{F} \cdot \boldsymbol{r}$$

a) Sketch the system and the relevant parameters.

b) Which force is acting on the particle? How do its equation of motion look like?

c) Verify that the energy is conserved.

d) Show that also the following quantity is a constant of motion,

$$I = \boldsymbol{F} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{L}) - \alpha \, \frac{\boldsymbol{F} \cdot \boldsymbol{r}}{|\boldsymbol{r}|} + \frac{1}{2} \, (\boldsymbol{F} \times \boldsymbol{r})^2$$

Here $\boldsymbol{L}$ is the angular momentum of the particle with respect to the origin of the coordinate system.

3.7 Further Reading

A comprehensive discussion of the flight of water bottle rockets has been given in Finney (2000), and it has been augmented by a discussion of subtle corrections involving the thermodynamic expansion of air in Gommes (2010).

4
Motion of Point Particles

In Chapter 3 we learned how to set up a physical model based on finding the forces acting on a body, and thus determining the acceleration of its motion. For a particle of mass m and position $\boldsymbol{q}$ Newton's second law relates its acceleration $\ddot{\boldsymbol{q}}$ to the force that is acting on the particle. In Chapter 2 we saw that the total force $\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ acting on the particle may depend on $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and t. The resulting relation between the acceleration and the force is called equation of motion of the particle.

Definition 4.1: Equation of Motion (EOM)

Newton's second law establishes a relation between the position $\boldsymbol{q}(t)$ of a particle of mass m, its velocity $\dot{q}(t)$, and acceleration $\ddot{q}(t)$,

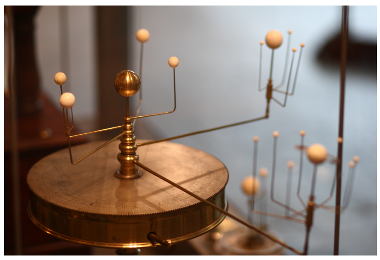
$$m\,\ddot{\boldsymbol{q}}(t) = \boldsymbol{F}(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t), t)$$

that is referred to as the *equation of motion* (EOM) of the particle.

The motion of N particles residing at the positions $\boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t) \in \mathbb{R}^D$ and interacting with each other, it amounts to $N\,D$ coupled equations

$$\begin{aligned} m\,\ddot{\boldsymbol{q}}_1(t) &= \boldsymbol{F}_1(\dot{\boldsymbol{q}}_1, \dots, \dot{\boldsymbol{q}}_N, \boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t), t) \\ \vdots \quad &= \quad \vdots \\ m\,\ddot{\boldsymbol{q}}_N(t) &= \boldsymbol{F}_N(\dot{\boldsymbol{q}}_1, \dots, \dot{\boldsymbol{q}}_N, \boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t), t) \end{aligned}$$

In the present chapter we will discuss approaches that will allow us to systematically find the solutions of EOMs. Moreover, we will explore what type of behavior is encountered for different types of initial conditions. At the end of this chapter we will discuss the motion of planets around the sun, moons around their planets, and will be able to figure out which rules determine the intricate trajectory of 'Oumuamua shown in Figure 3.9.



Mechanical planetarium used to teach astronomy at Harvard

4.1 Motivation and Outline: What are EOMs and ODEs?

From the mathematical point of view the equation of motion is an *ordinary differential equation* (ODE).

Definition 4.2: Ordinary Differential Equation (ODE)

An *ordinary differential equation (ODE)* of n^{th} order for a function $f(t)$ expresses the n^{th} derivative of the function, $f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$ as a function of time and the lower derivatives of the function, $f^{(n-1)}(t), \dots, f^{(1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$, $f^{(0)}(t) = f(t)$,

$$f^{(n)}(t) = F(f^{(n-1)}(t), \dots, f(t), t) \,.$$

Here, f and F may be scalar or vector valued functions.

Remark 4.1. The EOM for a particle at position $\boldsymbol{q} \in \mathbb{R}^3$ is a second order ODE where the second time derivative $\ddot{\boldsymbol{q}}(t)$ of the vector valued function $\boldsymbol{q}(t)$ (the position of the particle) is related to $\boldsymbol{F}/m$, which is a vector that depends on $\dot{\boldsymbol{q}}$, $\boldsymbol{q}$ and t; cf. Definition 4.1.

Remark 4.2. A differential equation is called an *ordinary differential equation*, when all derivatives are taken with respect to the same variable. When discussing the physics of waves, e. g. for the full description of Tsunami waves mentioned in Example 1.11, to deal with electromagnetic waves or gravitational waves, one has to deal with differential equations involving space and time derivatives. These type of equations are called *partial differential equations (PDE)*. In Leipzig they are addressed in the course "Theoretical Physics II".

Commonly, the forces in an EOM for a particle only depend on particle positions and velocities, and not explicitly on time, and not explicitly on time. The forces only depend on the particle configuration, and they will be the same irrespective of whether I measure them today or when the Corona crisis has become ancient history.

Definition 4.3: Autonomous Equations of Motion

An ODE is called *autonomous* when its right-hand side does not explicitly depend on time. In particular an autonomous EOM takes the form

$$m\, \ddot{\boldsymbol{q}}(t) = \boldsymbol{F}(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t)) \,.$$

The forthcoming discussion of ODEs makes use of the very important observation that every ODE can be stated as first order ODE in some abstract phase space. We introduce this idea for N particles with masses m_i, $i = 1 \cdots N$ that are moving in D dimensions. According to Definition 4.1 their motion is described by a system of $N\,D$ differential equations for the coordinates of the D dimensional vectors $\boldsymbol{q}_i = (q_{i,\alpha}, \alpha = 1 \cdots D)$

$$\ddot{q}_{i,\alpha} = \frac{1}{m_i}\, F_{i,\alpha}(\{\dot{\boldsymbol{q}}_i, \boldsymbol{q}_i\}_{i=1\cdots N}, t) \,, \quad i = 1 \cdots N \,, \;\; \alpha = 1 \cdots D$$

To avoid clutter in the equations we did not explicitly state here the time dependence of $\ddot{q}_{i,\alpha}(t)$, $\dot{q}_{i,\alpha}(t)$, and $q_{i,\alpha}(t)$.

By introducing the variables $\boldsymbol{v}_i = \dot{\boldsymbol{q}}_i$ the EOMs can be written as a set of $2\,D\,N$ first order ODEs

$$\begin{aligned}
\dot{q}_{i,\alpha} &= v_{i,\alpha} \\
\dot{v}_{i,\alpha} &= \frac{1}{m_i}\, F_{i,\alpha}(\{\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i\}_{i=1\cdots N}, t)
\end{aligned}$$

For an autonomous system this can be written in a more compact form by introducing the $2DN$ dimensional phase-space coordinate $\boldsymbol{\Gamma}$ and the flow $\boldsymbol{\mathcal{V}}$ as follows

$$\begin{aligned}
\boldsymbol{\Gamma} &= \big(q_{1,1} \cdots q_{1,D}\,,\; q_{2,1} \cdots q_{N,D}\,,\; \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\; \dot{q}_{2,1} \cdots \dot{q}_{N,D}\big) \\
\boldsymbol{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\; v_{2,1} \cdots v_{N,D}\,,\; \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\; \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\right) \\
\dot{\boldsymbol{\Gamma}} &= \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}) \qquad \text{for autonomous systems.}
\end{aligned}$$

Moreover, a non-autonomous system can always be expressed as an autonomous, first order ODE where $\boldsymbol{\Gamma}$ and $\boldsymbol{\mathcal{V}}$ denote points in a $2DN + 1$ dimensional phase space,

$$\begin{aligned}
\boldsymbol{\Gamma} &= \big(q_{1,1} \cdots q_{1,D}\,,\; q_{2,1} \cdots q_{N,D}\,,\; \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\; \dot{q}_{2,1} \cdots \dot{q}_{N,D}\,,\; t\big) \\
\boldsymbol{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\; v_{2,1} \cdots v_{N,D}\,,\; \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\; \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\,,\; 1\right) \\
\dot{\boldsymbol{\Gamma}} &= \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}) \qquad \text{for non-autonomous systems.}
\end{aligned}$$

In phase space, $\boldsymbol{\Gamma}$ denotes a point that characterizes the state of our system, and $\boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma})$ provides the *unique* direction and velocity of the temporal change of this state. In an approximation, that is accurate for sufficiently small Δt, we have

$$\boldsymbol{\Gamma}(t + \Delta t) \simeq \boldsymbol{\Gamma}(t) + \Delta t\; \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}(t))$$

In phase space the ODE therefore can be represented as a field of vectors $\boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma})$ that represent signposts signifying which direction a trajectory will take when it continues from this point, and how fast it will proceed.

Definition 4.4: Phase-Space Plot

A *phase-space plot* provides an overview of all solutions of an ODE by marking the direction of motion of the trajectories in phase space by arrows, and showing the evolution of a representative set of trajectories by solid lines. At times such a plot is therefore also denoted as the *phase-space portrait* of the solutions of an ODE.

Remark 4.3. For an autonomous system with a single DOF

$$\begin{aligned}
\dot{x}(t) &= v(t) \\
\dot{v}(t) &= m^{-1}\, F\big(v(t), x(t)\big)
\end{aligned}$$

the phase-space portrait is a two-dimensional plot with arrows $(v, F(v,x)/m$ at the positions (x, v) in the plane, and trajectories $v(x)$. One can only see the shape of the trajectories, and not their time dependence.

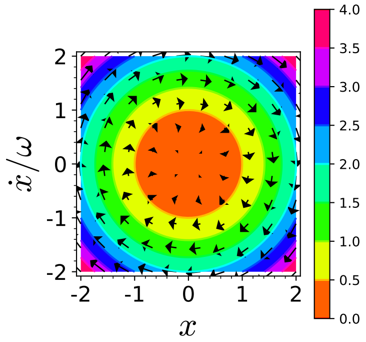


Figure 4.1: Color plot of contour lines of the energy of the harmonic oscillator, and the flow field of its EOM.

Example 4.1: Phase-space plot for the harmonic oscillator

The EOM of the harmonic oscillator is $\ddot{x}(t) = -\omega^2 x(t)$ where ω can be absorbed into the time scale by adopting the dimensionless time $\tau = \omega t$. Its dimensionless energy is then given by

$$E = \frac{v^2}{2} + \frac{x^2}{2} \quad \text{with} \quad \begin{cases} \dot{x} &= v \\ \dot{v} &= -x \end{cases}$$

The energy is conserved because

$$\frac{\mathrm{d}}{\mathrm{d}t} E = x\,\dot{x} + v\,\dot{v} = x\,v - v\,x = 0$$

Therefore trajectories in phase space amount to contour lines of the the energy function $E(x, v)$. This is shown in Figure 4.1 where the energy is marked by color coding and the direction of the flow is provided by arrows.

Outline

The forthcoming discussion in the present chapter will

i. provide a classification of ODEs with an emphasis on strategies to find solutions for specific initial conditions, and
ii. further discussion of phase-space plot used to characterize sets of solutions.

The methods will be introduced and motivated based on elementary physical problems that will serve as examples of particular relevance in physics.

4.2 *Free flight: Integrating ODEs*

We first discuss the motion of a single particle moving in a gravitational field that gives rise to the constant gravitational acceleration $\boldsymbol{g}$. Hence, the particle position $\boldsymbol{q}(t)$ obeys the EOM

$$\ddot{\boldsymbol{q}} = \boldsymbol{g} \tag{4.2.1}$$

The right hand side of this equation is constant. It neither depends on $\dot{\boldsymbol{q}}$, $\boldsymbol{q}$, nor explicitly on t. This has two remarkable consequences that we will exploit whenever possible.

4.2.1 *Decoupling of the motion of different DOF*

Each component q_α of $\boldsymbol{q}$ can be solved independently of the other DOF

$$\dot{q}_\alpha = g_\alpha$$

Rather than dealing with a vector-valued ODE, one can therefore solve D scalar ODEs which turns out to be a much simpler task.

Indeed, we will see in our further discussion that the solution of vector-valued ODEs will often proceed via a coordinate transformation that decouples the different DOF.

4.2.2 Solving ODEs by integration

The ODE, Equation (4.2.1), can be solved by integration

Algorithm 4.1: Integrating ODEs

An ODE for $\boldsymbol{f}(t)$ can be solved by *integration* when its right-hand side does not depend on on $\boldsymbol{f}(t)$ and its derivatives, i. e. when it takes the form

$$\dot{\boldsymbol{f}}(t) = \boldsymbol{g}(t)$$

For the initial condition $\boldsymbol{f}(t_0) = \boldsymbol{f}_0$ one can then expresses the solution of the ODE in terms of an integral,

$$\boldsymbol{f}(t) = \boldsymbol{f}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \dot{\boldsymbol{f}}(t') = \boldsymbol{f}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \boldsymbol{g}(t')$$

For an autonomous ODE, where $\boldsymbol{g}(t) = \boldsymbol{c} = \text{const}$, one thus obtains

$$\boldsymbol{f}(t) = \boldsymbol{f}_0 + \boldsymbol{c} \cdot (t - t_0)$$

The idea underlying the algorithm can be understood by reading the equations in reverse order and taking into account the substitution rule for integration,

$$\int_{t_0}^{t} \mathrm{d}t'\, g(t') = \int_{t_0}^{t} \mathrm{d}t'\, \dot{f}(t') = \int_{f(t_0)}^{f(t)} \mathrm{d}f = f(t) - f(t_0)$$

4.2.3 Integrating the EOM for free flight

For the free flight only the constant acceleration $\boldsymbol{g}$ due to gravity is acting on the particle such that $\ddot{\boldsymbol{q}}(t) = \boldsymbol{g}$. For the initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}(t_0) = \boldsymbol{v}_0$ Algorithm 4.1 provides the velocity

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}_0 + \int_{t_0}^{t} \mathrm{d}t' \boldsymbol{g} = \boldsymbol{v}_0 + \boldsymbol{g} \cdot (t - t_0)$$

This equation can be integrated again, providing the position of the particle

$$\begin{aligned}
\boldsymbol{q}(t) &= \boldsymbol{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \dot{\boldsymbol{q}}(t) = \boldsymbol{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \left(\boldsymbol{v}_0 + \boldsymbol{g}\,(t - t_0)\right) \\
&= \boldsymbol{q}_0 + \boldsymbol{v}_0 \int_{t_0}^{t} \mathrm{d}t' + \boldsymbol{g} \int_{t_0}^{t} \mathrm{d}t'\, (t - t_0) \\
&= \boldsymbol{q}_0 + \boldsymbol{v}_0\,(t - t_0) + \boldsymbol{g} \int_{0}^{t - t_0} \mathrm{d}t''\, t'' \\
&= \boldsymbol{q}_0 + \boldsymbol{v}_0\,(t - t_0) + \frac{1}{2}\,\boldsymbol{g}\,(t - t_0)^2
\end{aligned}$$

When we denote the direction anti-parallel to $\boldsymbol{g}$ as $z = q_1$, then

$$z(t) = q_1(t) = z(t_0) + v_z(t_0)\,(t - t_0) - \frac{g}{2}\,(t - t_0)^2 \tag{4.2.2a}$$

$$q_i(t) = q_i(t_0) + v_i(t_0)\,(t - t_0)\,, \quad \text{for } i > 1 \tag{4.2.2b}$$

It is illuminating to discuss these solutions form the perspective of non-dimensionalization and the evolution in phase space.

For $i > 1$ the EOM is $\ddot{q}_i = 0$. In phase space the direction field at (q_i, v_i) is then given by the vectors $(v_i(t_0), 0)$ pointing in horizontal direction, as shown in Figure 4.2(left). Moreover, for Equation (4.2.2b) we have $\dot{q}_i(t) = v_i(t_0) = \text{const}$ irrespective of $q_i(t)$. Therefore, the solutions take the form of horizontal lines. When introducing dimensionless units by adopting the velocity scale $v_i(t_0)$ one obtains

$$\hat{v}_i(t) = \frac{\dot{q}_i(t)}{\dot{q}_i(t_0)} = 1$$

For this problem all trajectories are identical up to a rescaling of the length and time units. By rescaling, all horizontal lines in the phase space can be mapped into the same dimensionless solution. From the point of view of the Buckingham-Pi Theorem 1.1 this is due to the fact that there are no dimensionless parameters in the solutions—not even due to the choice of initial conditions.

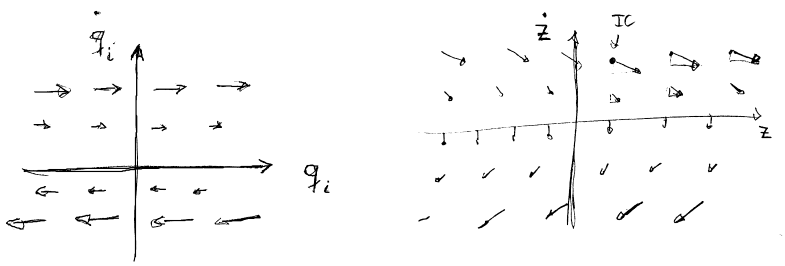


Figure 4.2: Phase-space flows for motion for free flight. (left) For direction perpendicular to g where there is no acceleration. The trajectories are horizontal lines. (right) For z anti parallel to g there is a constant acceleration $-g$. The trajectories take the form of parabola that are open to the left.

For Equation (4.2.2a) the arrows at position (z, v_z) in the phase space are directed to $(v, -g)$. For $v = 0$ they point straight down, for large v they point right and only a little bit down, and for large negative v they point left and only a little bit down, as marked in Figure 4.2(right). The phase-space trajectories are found by observing that $\dot{z} = v_z(t_0) - g\,(t - t_0)$ implies $t - t_0 = \big[v_z(t_0) - v_z\big]/g$ such that

$$\begin{aligned}
z &= z(t_0) + v_z(t_0)\,\frac{v_z(t_0) - v_z}{g} - \frac{\big[v_z(t_0) - v_z\big]^2}{2\,g} \\
&= z(t_0) + \frac{v_z(t_0) - v_z}{2\,g}\,\big[2\,v_z(t_0) - \big(v_z(t_0) - v_z\big)\big] && (4.2.3) \\
&= z(t_0) + \frac{v_z^2(t_0) - v_z^2}{2\,g} && (4.2.4)
\end{aligned}$$

As function of v_z these are parabola with a maximum at $v_z = 0$ and height $z_{\max} = z(t_0) + v_z^2(t_0)/2g$, as shown in Figure 4.2(right). In this case the EOM involves the constant g such that only one of the initial conditions can be absorbed into dimensionless units. For dimensionless units based on the velocity scale $v_z(t_0)$ and the length scale $v_z^2(t_0)/2\,g$ we have

$$\frac{z}{v_z^2(t_0)/2\,g} = I - \left(\frac{v_z}{v_z(t_0)}\right)^2 \quad \text{with} \;\; I = 1 + \frac{2\,g\,z(t_0)}{v_z^2(t_0)}$$

The trajectories in this dimensionless representation are shown in Figure 4.3. They all have the shape of a normal parabola, but the parabolas are shifted by the dimensionless constant I that is formed by the gravitational acceleration g, and the initial conditions $z(t_0)$ and $v_z(t_0)$.

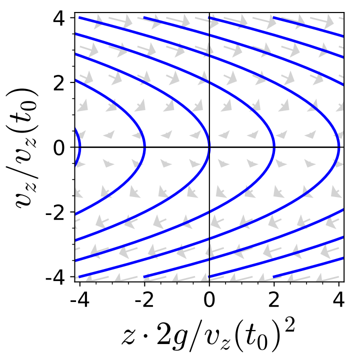


Figure 4.3: Dimensionless phase-space trajectories of a particle falling in the gravitational field without friction.

4.2.4 *Self Test*

Problem 4.1. **Estimating the depth of a pond**

You drop a stone into a pond and count n seconds till you hear it hit the water. How long a chord do you have to attach to your bucket to get up some water.

Problem 4.2. **Integrating the EOM for the flight of an Earth-bound rocket**

Integrate the EOM for rocket flight derived in Section 3.5,

$$\begin{aligned}\dot{V}_R(t) &= -g + \frac{a\,\rho\,v_f^2}{m + M_0 - a\,\rho\,v_f\,t}\\ \dot{z}(t) &= V_R(t)\end{aligned}$$

for a rocket that is launched with velocity v_0 at a height H_0, i. e. for the ICs

$$V_R(t_0) = v_0 \qquad \text{and} \qquad z(t_0) = H_0$$

How do the solutions Equations (3.5.1) and (3.5.2) change? Was there a way to anticipate the impact of the changing the initial height H_0?

Problem 4.3. **Alternative dimensionless units for trajectories with constant acceleration**

Discuss the shape of the trajectories that emerges when introducing dimensionless units based on the velocity scale $v_z(t_0)$ and the length scale $z(t_0)$ into Equation (4.2.4).

Hint: You will find parabolas as shown to in the margin. Discuss their shape and the position of their maximum.

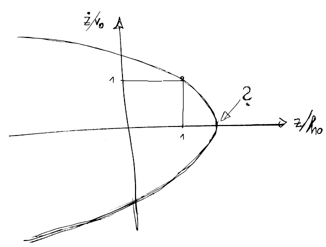


Figure 4.4: Sketch of the universal form of the free-flight trajectories in phase space, Equation (4.2.4).

4.3 *Settling with Stokes drag: Separation of variables*

The settling of a ball in a viscous medium can be described by the equations of motion

$$m\,\ddot{h}(t) = -m\,g - \mu\,\dot{h}(t)\,. \tag{4.3.1a}$$

Here $h(t)$ is the vertical position of the ball (height), g is the acceleration due to gravity, and the contribution $-\mu\,\dot{h}(t)$ describes Stokes friction, i. e. the viscous drag on the ball. It has the same form as the friction opposing the motion of the mine cart in Example 3.6.

The Stokes friction coefficient μ depends the viscosity of the fluid η and the geometry of the body. The viscosity $[\eta]$ of a fluid is measured in terms of $\mathrm{Pa} = \mathrm{kg/m\,s}$. For air and water it takes values of about $\eta_{\mathrm{air}} \simeq 2 \times 10^{-5}\,\mathrm{kg/m\,s}$, and $\eta_{\mathrm{water}} \simeq 1 \times 10^{-3}\,\mathrm{kg/m\,s}$, respectively. The size of the ball will be given by its radius R. Hence, dimensional analysis implies that

$$\mu \propto R\,\eta$$

For a sphere of radius R the proportionality constant takes the value of 6π.

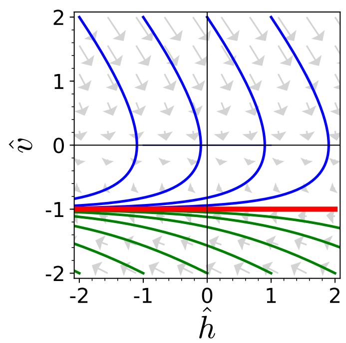


Figure 4.5: Dimensionless phase-space trajectories of a particle subjected to a constant acceleration g and Stokes drag.

This problems involves the parameters g, μ and m that will absorbed into dimensionless units by introducing the dimensionless units for height $\hat{h} = h\,\mu^2/m^2\,g$, velocity $\hat{v} = \dot{h}\,\mu/m\,g$, and time $\tau = (t - t_0)\,\mu/m$. In these units the EOM takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\hat{h}(\tau) = -1 - \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{h}(\tau) \quad \Leftrightarrow \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{h} & = \hat{v} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{v} & = -1 - \hat{v} \end{cases} \tag{4.3.1b}$$

The corresponding phase-space plot is shown in Figure 4.5. For positive (i. e. upwards) velocities the resulting direction field in phase space point to the lower right, and for $\hat{v}$ it points straight down. However, the arrows are steeper than for the case without friction, Figure 4.3. For $\hat{v} > 0$ the trajectories in the two cases look similar, but with friction they follow curves that are broader than the parabola for the frictionless fall. For downwards, the flows differ qualitatively: Trajectories started with zero velocity never cross the $\hat{v} = -1$ line, and trajectories that are started with a speed larger than 1 are no longer accelerated by gravity, but slowed down by friction until they also reach their terminal velocity -1 that is marked by a red line.

After having reached this qualitative insight into the dynamics, we will look now for the explicit solution of the EOM. Equation (4.3.1) can be integrated once, yielding an ODE for the settling velocity starting from a height $\hat{h}_0$ with velocity $\hat{v}_0$,

$$\dot{h}(\tau) = \hat{v}_0 - \tau - \left(\hat{h}(\tau) - \hat{h}_0\right)$$

This equation can not be solved by integration, employing Algorithm 4.1, because its right-hand side explicitly depends on the function $h(t)$ that must still be determined as solution of the ODE. It is a better strategy in this case to adopt another solution strategy.

4.3.1 Solving ODEs by separation of variables

In the case at hand the ODE (4.3.1a) can be interpreted as a first order ODE for the settling velocity $v = \dot{h}$ where $\dot{v}$ is provided as a function of only v. Such an ODE is best solved by separation of variables.

Algorithm 4.2: Separation of variables

A one-dimensional first-order ODE of the form

$$\dot{f}(t) = g(f(t))\, h(t)$$

can be solved by *separation of variables*. For the initial condition $f(t_0) = f_0$ one obtains then

$$\int_{t_0}^{t} \mathrm{d}t'\, h(t') = \int_{t_0}^{t} \mathrm{d}t'\, \frac{\dot{f}(t')}{g(f(t'))} = \int_{f_0}^{f(t)} \mathrm{d}f\, \frac{1}{g(f)}$$

which provides the solution in terms of two integrals.

Remark 4.4. Let us assume that we find the antiderivatives $H(t)$ with $\mathrm{d}H/\mathrm{d}t = h(t)$ as well as $A(f)$ with $\mathrm{d}A/\mathrm{d}f = 1/g(f)$ and inverse $I(f)$, i.e. $I(A(f)) = f$. Then separation of variables provides the explicit solution

$$\begin{aligned} H(t) - H(t_0) &= A(f(t)) - A(f(t_0)) \\ \Rightarrow \quad f(t) &= I\big(H(t) - H(t_0) + A(f(t_0))\big) \end{aligned}$$

Remark 4.5. Often the integrals can be performed but the inverse $I(f)$ can not be given in a closed form. If one can find the inverse of $H(t)$, i.e. a function $J(H)$ with $J(H(t)) = t$ then the solution can still be given in the (rather unusual) explicit form

$$t = J\big(A(f) - A(f(t_0)) + H(t_0)\big)$$

This is always possible for autonomous ODEs, i.e. in particular for Equation (4.3.1a) with $f(t) = \dot{h}(t)$.

Remark 4.6. When neither of the inverse functions are known, then the solution can only be stated as an implicit equation

$$H(t) - A(f) = H(t_0) - A(f(t_0)) = \text{const}$$

Hence, the solutions amount to the contour lines of the function $G(t, f) = H(t) - A(f)$ that is plotted to this end as function of the two variables (t, f).[1]

[1] Plotting of contour lines is supported by all scientific plot programs. In Gnuplot it is facilitated via the "set implicit" option for a 2d plot command "plot", or by using "set contour" together with a 3d plot called by "splot". In Sage there are the commands 'plot_implicit()' and 'contour_plot()'. In Python with Matplotlib there is 'matplotlib.pyplot.contour()'.

4.3.2 Solving the EOM for settling with Stokes drag

For Equation (4.3.1a) we will now derive the velocity $v(t) = \dot{h}(t)$ for an initial velocity v_0 by applying Algorithm 4.2. In order to simplify notations we perform the derivation in dimensionless units, Equation (4.3.1b), and introducing the physical variables in the end. Separation of variables provides that for a particle with initial velocity $\hat{v}_0$

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w \, \frac{1}{1+w} = -\ln \frac{1+\hat{v}(\tau)}{1+\hat{v}_0}$$

$$\Leftrightarrow \quad \hat{v}(\tau) = -1 + \left(\hat{v}_0 + 1\right) \mathrm{e}^{-\tau} \tag{4.3.2}$$

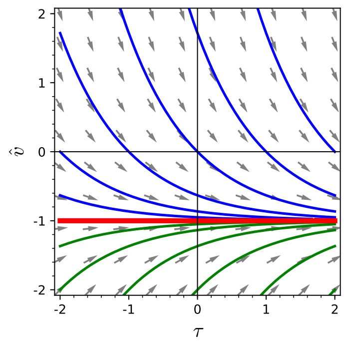


Figure 4.6: Sketch of $w(\tau)$ as obtained in Equation (4.3.2).

The solutions are shown in Figure 4.6. Stokes drag entails that for large times, $\tau \gg 1$, the ball is sinking with the constant Stokes velocity that takes the value -1 in our dimensionless units. Due to $-1 = \hat{v}_\infty = \mu v_\infty / m\, g$ this implies $v_\infty = -m\, g/\mu$ in terms of the physical units.

The position of the sphere can be obtained by integrating Equation (4.3.2) for $\hat{v}(\tau) = \mathrm{d}\hat{h}/\mathrm{d}\tau$ with initial condition $\hat{h}_0$,

$$\begin{aligned}\hat{h}(\tau) &= \hat{h}_0 + \int_0^\tau \mathrm{d}\tau \, \frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \hat{h}_0 + \int_0^\tau \mathrm{d}\tau \left(-1 + \left(\hat{v}_0 + 1\right) \mathrm{e}^{-\tau}\right) \\ &= \hat{h}_0 - \tau + \left(\hat{v}_0 + 1\right) \; \left(1 - \mathrm{e}^{-\tau}\right) \end{aligned} \tag{4.3.3a}$$

or in terms of physical units

$$h(t) = h_0 - v_\infty \, (t - t_0) + \frac{m}{\mu} \, (v_0 - v_\infty) \; \left[1 - \exp\left(-\frac{\mu}{m}(t - t_0)\right)\right] \tag{4.3.3b}$$

4.3.3 *Relation to free fall*

It is instructive to explore how the evolution with Stokes friction is related to the free flight $h_f(t) = h_0 + v_0(t - t_0) - g\,(t - t_0)^2$ obtained in Section 4.2. This can most effectively be done by Taylor expansion of Equation (4.3.3) for small τ, and subsequently expressing the result in physical units.

Definition 4.5: Taylor expansion

The *Taylor expansion* to order N provides an approximation of a function $f(x)$ at a position x_0. It is obtained by matching the first N derivatives of the function and of a polynomial of order N that represents the Taylor approximation (or *Taylor approximation*),

$$f(x) \simeq \sum_{n=0}^{N} \left. \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} \right|_{x=x_0} \frac{(x - x_0)^n}{n!}$$

Remark 4.7 (Leading-order Taylor expansion). The first-order, or leading-order Taylor expansion is a linear function $t(x) = t_0 + t_1 x$ with coefficients $t_0 = f(x_0)$ and $t_1 = f'(x_0)$. Hence, we have $t(x_0) = t_0 = f(x_0)$ and $t'(x) = t_1 = f'(x_0)$. This is a tangent to the function $f(x)$ that approximates f at x_0 by having the same functional value and slope. Examples for the sine function are shown in Figure 4.7.

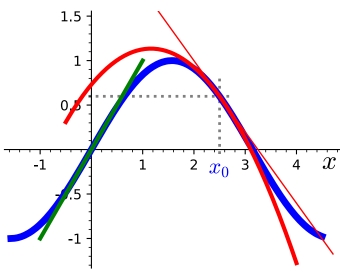


Figure 4.7: The leading order and second order Taylor approximations of the sine function at the origin (green) and at the position $x_0 = 2.5$.

Remark 4.8 (Second-order Taylor expansion). The second-order Taylor expansion is a quadratic function $t(x) = t_0 + t_1 x + t_2 x^2$ with coefficients $t_0 = f(x_0)$, $t_1 = f'(x_0)$, and $t_2 = f''(x_0)/2$. As for the first-order approximation, we have $t(x_0) = t_0 = f(x_0)$ and $t'(x) = t_1 = f'(x_0)$. Moreover, in this case we also have $t''(x) = 2\,t_2 = f'(x_0)$. Examples for the sine function are shown in Figure 4.7.

Example 4.2: Taylor approximations of the sine function

For $\sin x$ the even derivatives vanish at the origin, and the odd $2n-1$ derivative amounts to -1^n. Hence, the first few terms of the Taylor expansion at the origin are given by

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

At the origin the first and second order Taylor approximation agree, as shown by the green line in Figure 4.7.
For the sine-function the expansion at a position x_0 is given by

$$sinx \simeq \sin(x_0)\,\left[1 - \frac{(x-x_0)^2}{2!} + \frac{(x-x_0)^4}{4!} - \frac{(x-x_0)^6}{6!} + \cdots\right]$$
$$+ \cos(x_0)\,\left[(x-x_0) - \frac{(x-x_0)^3}{3!} + \frac{(x-x_0)^5}{5!} - \cdots\right]$$

The red lines in Figure 4.7 show the first-order (thin red line) and the second order (thick red line) approximation for $x_0 = 2.5$. The second order approximation remains closer to the sine-function for a bit longer than the linear first-order approximation.

Example 4.3: Taylor approximations of the exponential function

The derivatives of the exponential function $f(x) = \exp(ax)$ amount to $f^{(n)}(x) = a^n f(x)$ such that its expansion at a position x_0 is given by

$$\mathrm{e}^{a\,x} \simeq \mathrm{e}^{a\,x_0}\,\left[1 + a\,(x-x_0) + \frac{a^2\,(x-x_0)^2}{2} + \cdots + \frac{a^n\,(x-x_0)^n}{n!} + \cdots\right]$$

For $x_0 = 0$ this simplifies to

$$\mathrm{e}^{a\,x} = \sum_{n=0}^{\infty} \frac{(a\,x)^n}{n!} \simeq 1 + a\,x + \frac{(a\,x)^2}{2} + \cdots$$

Based on the Taylor expansion of the exponential function $\mathrm{e}^{-\tau} = \sum_{n=0}^{\infty}(-\tau)^n/n!$ we find for Equation (4.3.3)

$$\begin{aligned}\hat{h}(\tau) &= \hat{h}_0 - \tau + (\hat{v}_0 + 1)\,\left(\tau - \frac{\tau^2}{2}\frac{\tau^3}{6} - \cdots\right)\\ &= \hat{h}_0 + \hat{v}_0\,\tau - (\hat{v}_0 + 1)\,\frac{\tau^2}{2}\,\left(1 - \frac{\tau}{3} + \cdots\right)\end{aligned}$$

The solution with physical units is obtained by substituting $\hat{h} = \mu^2 h/m^2 g$, $\hat{h}_0 = \mu^2 h_0/m^2 g$, $\tau = \mu\,(t - t_0)/m$, and $\hat{v}_0 = \mu\,\dot{h}(t_0)/m\,g$. Hence,

$$h(t) = h_0 + v_0\,(t-t_0) - \frac{\mu}{m}\left(v_0 + \frac{m\,g}{\mu}\right)\frac{(t-t_0)^2}{2}\left(1 - \frac{\mu\,(t-t_0)}{3\,m} + \cdots\right)$$
$$= h_0 + v_0\,(t-t_0) - \frac{g}{2}\,(t-t_0)^2\left(1 - \frac{v_0}{v_\infty}\right)\left(1 - \frac{\mu\,(t-t_0)}{3\,m} + \cdots\right)$$

This implies that Stokes friction provides a small corrections to the free flight if the initial velocity is small as compared to the asymptotic velocity of free flight, $|v_0| \ll v_\infty = m\,g/\mu$. Further, one must restricts the attention to times that are small as compared to the time scale m/μ where the asymptotic velocity is reached. Equation (4.3.2) implies that this amounts to situations where the velocity $|v(t)|$ is small as compared to the Stokes settling speed v_∞. This is discussed now for two concrete cases:

Example 4.4: Stokes friction for a steel ball

A steel ball with a diameter of 1 cm has a mass of about

$$m = \frac{4\pi}{3}\,2\times10^3\,\mathrm{kg/m^3}\,\frac{1\times10^{-6}\,\mathrm{m^3}}{8} \simeq 1\times10^{-3}\,\mathrm{kg}$$

In air it will reach a terminal velocity of about

$$v_{\text{air}} = \frac{m\,g}{\mu_{\text{air}}} = \frac{3\,m\,g}{2\,\eta_{\text{air}}\,R} = \frac{3\times1\times10^{-3}\,\mathrm{kg}\,10\,\mathrm{m/s^2}}{2\times2\times10^{-5}\,\mathrm{kg/m\,s}\,1\times10^{-2}\,\mathrm{m}}$$
$$\simeq 7.5\times10^4\,\mathrm{m/s}$$

Saturation to this velocity occurs on time scales

$$t_{\text{air}} = \frac{m}{\mu_{\text{air}}} = \frac{m}{\eta_{\text{air}}\,R} = \frac{1\times10^{-3}\,\mathrm{kg}}{2\times10^{-5}\,\mathrm{kg/m\,s}\,1\times10^{-2}\,\mathrm{m}} = 5\times10^3\,\mathrm{s}$$

and this time the bullet will have dropped by a distance $g\,t_c^2/2 = 2.5\times10^7\,\mathrm{m}$ which is much more than the thickness of the atmosphere. We conclude that Stokes friction is not relevant for the motion of a bullet in air.
Even in water, where the viscosity is larger by a factor of 50, we will have

$$v_{\text{water}} = \frac{3\,m\,g}{2\,\eta_{\text{water}}\,R} = \frac{3\times1\times10^{-3}\,\mathrm{kg}\,10\,\mathrm{m/s^2}}{2\times1\times10^{-3}\,\mathrm{kg/m\,s}\,1\times10^{-2}\,\mathrm{m}}$$
$$\simeq 1.5\times10^3\,\mathrm{m/s}$$

Saturation to this velocity occurs on time scales

$$t_{\text{water}} = \frac{m}{\eta_{\text{water}}\,R} = \frac{1\times10^{-3}\,\mathrm{kg}}{1\times10^{-3}\,\mathrm{kg/m\,s}\,1\times10^{-2}\,\mathrm{m}} = 100\,\mathrm{s}$$

and this time the bullet will have dropped by a distance $g\,t_{\text{water}}^2/2 = 5\times10^4\,\mathrm{m}$ which is deeper than the deepest point in our Oceans.

Example 4.5: Stokes friction for sperms

Sperms are cells equipped with cilia that allow them to swim towards the egg for fertilization. They have a characteristic size L of a few micrometers and they swim in an environment that is approximated here as water. Their mass is of the order of $m_{\text{sperms}} = \rho_{\text{water}}\, L^3$. In this case their asymptotic speed is reached at a time scale

$$\begin{aligned} t_{\text{spermium}} &= \frac{m_{\text{spermium}}}{\mu_{\text{spermium}}} = \frac{\rho_{\text{water}}\, L^2}{\eta_{\text{water}}} \\ &= \frac{1 \times 10^3\,\text{kg/m}^3\; 1 \times 10^{-12}\,\text{m}^2}{1 \times 10^{-3}\,\text{kg/m s}} = 1 \times 10^{-6}\,\text{s} \end{aligned}$$

Stokes friction plays a major role for their swimming. See Purcell (1977) for more details.

4.3.4 *Self Test*

Problem 4.4. **Solving ODEs by separation of variables**

Determine the solutions of the following ODEs

a) $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$ such that $y(\pi/4) = 0$

b) $\dfrac{dy}{dx} = \dfrac{3\,x^2\,y}{2y^2 + 1}$ such that $y(0) = 1$

c) $\dfrac{dy}{dx} = -\dfrac{1 + y^3}{x\,y^2\,(1 + x^2)}$ such that $y(1) = 2$

Problem 4.5. **Taylor approximations of the Cosine function**

Find the Taylor approximation for the cosine function

a) analogously to the discussion in Example 4.2, and

b) based on Euler's equation $e^{ix} = \cos x + i \sin x$.
Hint: Insert $a = i$ into the expansion provided in Example 4.3, and collect real and imaginary parts.

Problem 4.6. **Stopping distance on the water**

A yacht of mass 750 kg is sailing on the sailing into the harbor with a speed of 6 m/s. At this moment it is experiencing a friction force of 900 N. At time $t = 0$ the skipper switches off the motor such that only the friction is acting on the boat. Let the water resistance be proportional to the speed.

a) How long will it take till the yacht has come to rest?

b) How long will if take till the speed has been reduced to 1.5 m/s and which distance has the yacht traversed till that time?

Problem 4.7. **Free fall with viscous friction.**

In Equation (4.3.3) we derived the time evolution of the height $h(t)$ of a ball that is falling a gravitational field and subjected to Stokes drag.

a) Make a plot of $\hat{h}(\tau)$ as function of τ, where you compare the evolution of trajectories that start with different initial velocity $\hat{v}_0$ from the same height $\hat{h}_0 = 0$.

b) Make a plot of $h(t)$ as function of t, where you compare the evolution of trajectories that start with the same initial velocity v_0 from the same height h_0, but are subjected to a different drag μ (for instance because they have different radius).

4.4 *Worked example: Free flight with turbulent friction*

In Example 4.4 we reached the puzzling conclusion that — for all physically relevant parameters — Stokes friction plays no role for the motion of a steel ball in air and water. On the other hand, we know from experience that friction arises to the very least for large velocities, like for gun shots. This apparent contradiction is resolved by observing that the drag is not due to Stokes drag. Rather for most settings in our daily live friction arises because the motion of the fluid around the considered object goes turbulent, as anticipated in Problem 3.3. A ball of mass m, radius R, and mass density $\rho_{\text{ball}} = 3m/4\pi R^3$ that is moving with speed v through a fluid of mass density ρ_{fluid} will experience a turbulent drag force of modulus

$$F_D = m \, \frac{\rho_{\text{fluid}} \, C_D}{8 \, \rho_{\text{ball}} \, R} \, v^2 = m \, \kappa \, v^2$$

Here, C_D is a dimensionless number that typically takes values between 0.5 and 1. A very beautiful description of the physics of this equations has been provided in an instruction video by the NASA (click here to check it out).

To address motion affected by turbulent drag we measure time in units of $(\kappa g)^{-1/2}$ and velocity in units of g/κ. The dimensionless velocity $\hat{v}(\tau)$ will then obey the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \, \hat{h}(\tau) = \hat{v} \tag{4.4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \, \hat{v}(\tau) = -1 - \hat{v}^2(\tau) \, \operatorname{sign}\big(\hat{v}(\tau)\big) \tag{4.4.2}$$

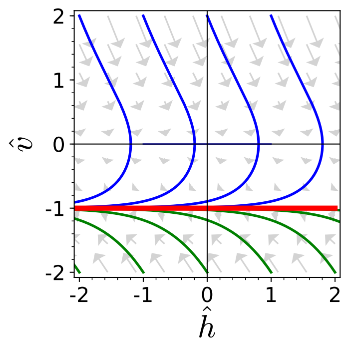


Figure 4.8: Dimensionless phase-space trajectories of a particle subjected to a constant acceleration g and turbulent drag.

The phase-space flow for this EOM is shown in Figure 4.8. It looks similar to the one for Stokes drag, with the important difference that the change of velocity grows much faster for large $|\hat{v}|$. For $\hat{v} > 0$ this gives even rise to an inflection point where the curvature of the trajectories indicated by blue lines crosses from concave to convex.

The equation for the velocity can again be solved by separation of variables,

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w \, \frac{1}{1 + w^2 \operatorname{sign}(w)}$$

In order to deal with the sign-function, we deal with the integral separately for four types of initial conditions in either of the intervals $\{(-\infty, -1), [-1, -1], (-1, 0], [0, \infty)\}$.

First we consider the initial condition where $\hat{v}_0 = -1$. In that case $\frac{\mathrm{d}}{\mathrm{d}\tau} w(\tau) = 0$ such that

$$\hat{v}(\tau) = -1 \qquad \text{for} \quad \hat{v}_0 = -1 \tag{4.4.3a}$$

Next we consider initial conditions where $\hat{v}_0 < 0$, but $\hat{v}_0 \neq 0$. In this case

$$\tau = \int_{\hat{v}_0}^{v(\tau)} \frac{\mathrm{d}w}{1 - w^2} = \frac{1}{2} \ln \left(\frac{1 - \hat{v}(\tau)}{1 + \hat{v}(\tau)} \cdot \frac{1 + \hat{v}_0}{1 - \hat{v}_0} \right)$$

$$\Leftrightarrow \quad \hat{v}(\tau) = \begin{cases} -\tanh(\tau - \operatorname{atanh} \hat{v}_0) & \text{for} \quad -1 < \hat{v}_0 \leq 0 \\ -\operatorname{cotanh}(\tau - \operatorname{acotanh} \hat{v}_0) & \text{for} \quad -1 > \hat{v}_0 \end{cases} \tag{4.4.3b}$$

Finally, we consider the case $\hat{v}_0 > 0$. We expect in that case that the particle moves up, $\hat{v}(\tau) > 0$, till some time τ_c, and then it start falling due to the action of gravity. However, in that case its velocity heads down with $-1 < \hat{v}_0 \leq 0$ such that it must follow the solution $\hat{v}(\tau) = -\tanh(\tau - \tau_c)$ obtained in Equation (4.4.3b). For $\tau < \tau_c$ we find

$$\tau = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w \, \frac{1}{1 + w^2} = -\arctan(\hat{v}(\tau)) + \arctan(\hat{v}_0)$$

$$\Leftrightarrow \quad \hat{v}(\tau) = -\tan(\tau - \arctan(\hat{v}_0))$$

such that

$$\hat{v}(\tau) = \begin{cases} -\tan(\tau - \tau_c) & \text{for} \quad \hat{v}_0 > 0 \wedge \tau < \tau_c = \arctan(\hat{v}_0) \\ -\tanh(\tau - \tau_c) & \text{for} \quad \hat{v}_0 > 0 \wedge \tau \geq \tau_c = \arctan(\hat{v}_0) \end{cases} \tag{4.4.3c}$$

A solution Equation (4.4.3) that passes through the origin and another one through $(0, -2)$ are shown in Figure 4.9.

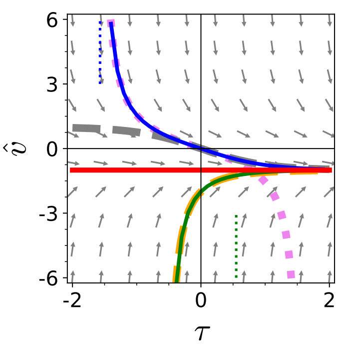


Figure 4.9: Solutions Equation (4.4.3) of Equation (4.4.1).

4.4.1 *Range of applicability*

Turbulent friction applies whenever

$$\mu \, |v| \lesssim m \, \kappa \, v^2 \quad \Leftrightarrow \quad |v| \gtrsim v_c = \frac{\mu}{m\kappa} \simeq \frac{\eta_{\text{fluid}}}{\rho_{\text{fluid}} \, R}$$

For the 1 cm steel ball considered in Example 4.4 the cross-over velocity v_c yields

$$v_c = \begin{cases} \dfrac{2 \times 10^{-5} \, \text{kg/m s}}{1 \, \text{kg/m}^3 \times 1 \times 10^{-2} \, \text{m}} = 2 \, \text{mm/s} & \text{for air} \\ \dfrac{1 \times 10^{-3} \, \text{kg/m s}}{1 \times 10^3 \, \text{kg/m}^3 \times 1 \times 10^{-2} \, \text{m}} = 0.1 \, \text{mm/s} & \text{for water} \end{cases}$$

Moreover, the characteristic time for turbulent drag is

$$t_c = (\kappa g)^{-1/2} = \sqrt{\frac{\rho_{\text{ball}}\, R}{\rho_{\text{fluid}}\, g}}$$
$$= \begin{cases} \sqrt{\frac{2\times 10^3\,\text{kg/m}^3 \times 1\times 10^{-2}\,\text{m}}{1\,\text{kg/m}^3 \times 10\,\text{m/s}^2}} \simeq 1.4\,\text{s} & \text{for air} \\ \sqrt{\frac{2\times 10^3\,\text{kg/m}^3 \times 1\times 10^{-2}\,\text{m}}{1\times 10^3\,\text{kg/m}^3 \times 10\,\text{m/s}^2}} \simeq 0.04\,\text{s} & \text{for water} \end{cases}$$

As a consequence, one may safely assume that Stokes friction is always negligible for the steel ball. Either friction may be neglected or turbulent friction must be considered.

4.4.2 *Self Test*

Problem 4.8. **Turbulent friction.**

Assume that the Earth atmosphere gives rise to the same turbulent drag, irrespective of height.

a) What is the maximum time after which a steel ball that is shot up with vertical velocity v_0 will hit the ground?

b) Does it make a noticeable difference when you require that v_0 must not surpass the speed of light $c = 3 \times 10^8\,\text{m/s}$?

Problem 4.9. **Free fall with turbulent friction.**

In Equation (4.4.3) we derived the velocity of a ball that is accelerated by gravity and slowed down by turbulent drag.

a) How will the height $\hat{h}$ of the trajectories evolve for large times τ?

b) Determine the full time dependence of the dimensionless height $\hat{h}$ by solving the ODE $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{h} = \hat{v}$.
Hint: Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\cos\tau = -\tan\tau \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\cosh\tau = \tanh\tau \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sinh\tau = \text{cotanh}\tau$$

c) Make a plot of $\hat{h}(\tau)$ as function of τ, where you compare the evolution of trajectories that start with different initial velocity $\hat{v}_0$ from the same height $\hat{h}_0 = 0$.

d) Insert the definitions of the dimensionless units in order to find the solutions for the velocity $v(t)$ and height $h(t)$ in physical units.

4.5 *Particle suspended from a spring: Linear ODEs*

There are two forces acting on a particle is suspended from a spring: the gravitational force $-m\,g$ and the spring force $-k\,z(t)$

where $z(t)$ measures the displacement of the spring from its rest position. Hence, the EOM of the particle takes the form

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) \tag{4.5.1}$$

This equation can neither be integrated directly, because its right hand side depends on $z(t)$, nor can it be solved by separation of variables, because its right hand side depends on $z(t)$ rather than only on $\dot{z}(t)$. It falls into the very important class of *linear ODEs*.

Definition 4.6: Linear ODEs

An ODE is called a *linear ODE* when $z(t)$ and its derivatives only appear as linear terms in the ODE. Hence, an N^{th} order linear ODEs for $z(t)$ takes the general form

$$I(t) = z^{(N)}(t) + c_{N-1}(t)\,z^{(N-1)}(t) + \cdots + c_0(t)\,z(t)$$

The functions $I(t)$, $c_\nu(t)$, $\nu = 0 \cdots N-1$, are called the coefficients of the linear ODE. When they do not depend on time we speak of a linear ODE with *constant coefficients*. In particular, $I(t)$ is called *inhomogeneity*; when it vanishes the ODE is called *homogeneous*.

Example 4.6: Particle suspended from a spring

Equation (4.5.1) is an inhomogeneous second-order linear ODE with the constant coefficients $f_0 = k$, $f_1 = 0$, and inhomogeneity $I = m\,g$.

Remark 4.9. An N^{th}-order linear ODE where the coefficient in front of the N^{th} derivative takes the value $c_N \neq 1$ can be stated in the form given in Definition 4.6 by division with c_N.

Example 4.7: Damped harmonic oscillator

The harmonic oscillator with damping γ and spring constant k

$$m\,\ddot{x}(t) = -m\,\gamma\,\dot{x}(t) - k\,x(t)$$

is described by a homogeneous second order, linear ODE with the constant coefficients $k_1 = \gamma$ and $k_0 = k/m$.

4.5.1 *Solving linear ODEs with constant coefficients*

Linear ODEs with constant coefficients are solved as follows

Algorithm 4.3: Linear ODEs with constant coefficients

An N^{th}-order linear ODE with constant coefficients,

$$I = \sum_{\nu=0}^{N} c_\nu\, f^{(\nu)}(t)$$

can be recast into a homogeneous ODE by considering $h(t) = f(t) - I/c_0$, which is a solution of the corresponding homogeneous, linear ODE

$$0 = \sum_{\nu=0}^{N} c_\nu \, h^{(\nu)}(t)$$

Its solutions can be written as

$$h(t) = \sum_{k=1}^{N} A_k \, \mathrm{e}^{\lambda_k \, t}$$

where the numbers λ_k, $k = 1 \cdots N$ are the N distinct roots of the characteristic polynomial

$$0 = \sum_{\nu=0}^{N} c_\nu \, \lambda^\nu$$

and the amplitudes A_k, $k = 1 \ldots N$ must be chosen such that $f(t) = I + c_0 \, h(t)$ obeys the initial conditions

$$\begin{aligned} f(t_0) &= \frac{I}{c_0} + \sum_{k=1}^{N} A_k \, \mathrm{e}^{\lambda_k \, t_0} \\ f^{(1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k \, \mathrm{e}^{\lambda_k \, t_0} \\ \vdots &= \vdots \\ f^{(N-1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k^{N-1} \, \mathrm{e}^{\lambda_k \, t_0} \end{aligned}$$

The idea underlying this algorithm is founded on three insights:

- the solutions of a homogeneous linear ODE form a vector space,
- $\exp(\lambda t)$ is a solution of the ODE iff it is a root of the characteristic polynomial, and
- the functions $\{\exp(\lambda_i t), i = 1, \ldots, N\}$ form a basis of the vector space.

The proof will be provided in Problem 4.13.

Remark 4.10. When the polynomial only has $M < N$ distinct roots the set of functions $\{\exp(\lambda_i \, t), i = 1, \ldots, M\}$ is missing $N - M$ elements to form a basis for the space of solutions. The set is augmented then by functions of the form $t \, \exp(\kappa \, t)$ for double roots, $t^2 \, \exp(\kappa \, t)$ for triple roots, etc. In this course we only deal with second order ODEs, where at most double roots arise. The solution strategy for that case will be discussed in Section 4.5.3.

4.5.2 Solving the ODE for the mass suspended from a spring

For Equation (4.5.1) this implies that $h(t) = z(t) + mg/k$ with

$$0 = \ddot{h}(t) + \frac{k}{m}\, h(t) \tag{4.5.2}$$

such that we obtain

$$\lambda_\pm = \pm\sqrt{\frac{k}{m}} = \pm\omega \qquad \text{as solution of} \quad 0 = \lambda^2 + \frac{k}{m}$$

Consequently, the motion of the spring is described by

$$z(t) = -\frac{mg}{k} + A_+\, \mathrm{e}^{\omega\,(t-t_0)} + A_-\, \mathrm{e}^{-\omega\,(t-t_0)}$$

This is a real-valued function if and only if A_+ and A_- are canonically conjugated complex numbers, such that we can write $A_\pm = A\, \exp(\pm\mathrm{i}\,\varphi)/2$ with $A \in \mathbb{R}$. As a consequence of $\cos x = (\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})/2$ we then obtain

$$z(t) = -\frac{mg}{k} + A\; \cos\bigl(\varphi + \omega\,(t - t_0)\bigr) \tag{4.5.3a}$$

where A and φ must be fixed based on the initial conditions

$$\begin{aligned} z(t_0) &= -\frac{mg}{k} + A\; \cos(\varphi) \\ \dot{z}(t_0) &= \qquad -\omega\, A\; \sin(\varphi) \end{aligned}$$

or

$$A^2 = \left(z(t_0) + \frac{mg}{k}\right)^2 + \frac{\dot{z}^2(t_0)}{\omega^2} \qquad \text{and} \qquad \varphi = \arcsin\left(\frac{\dot{z}(t_0)}{\omega\, A}\right) \tag{4.5.3b}$$

4.5.3 Solution for the damped harmonic oscillator

The damped harmonic oscillator is described by the linear EOM

$$0 = \ddot{x}(t) + \gamma\, \dot{x}(t) + \frac{k}{m}\, x(t) \qquad \text{with} \quad \gamma,\, k,\, m \in \mathbb{R}_+ \,. \tag{4.5.4}$$

It characteristic polynomial

$$0 = \lambda^2 + \gamma\,\lambda + \frac{k}{m}$$

has the solutions

$$\lambda_\pm = -\frac{1}{2}\,\left(\gamma \pm \sqrt{\gamma^2 - 4\,k/m}\right)$$

Here λ_+ and λ_- can either be both real, a pair of complex conjugated numbers, or we have to deal with the case $\gamma^2 = 4\,k/m$ where there only is a single root. We treat the cases one after the other.

1. Two real roots

In this case $\gamma^2 < 4\,k/m$ such that $\lambda_\pm \in \mathbb{R}_-$. The motion of the oscillator is described by

$$x(t) = A_+\, \mathrm{e}^{\lambda_+\,(t-t_0)} + A_-\, \mathrm{e}^{\lambda_-\,(t-t_0)}$$

which is a real-valued function for amplitudes $A_\pm \in \mathbb{R}$. The solution for the initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$ is then found by solving the equations

$$\left.\begin{aligned} x_0 &= A_+ + A_- \\ v_0 &= A_+\,\lambda_+ + A_-\,\lambda_- \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} A_+ &= m\,(x_0\,\lambda_- - v_0)/\sqrt{\gamma^2 - 4\,k/m} \\ A_- &= -m\,(x_0\,\lambda_+ - v_0)/\sqrt{\gamma^2 - 4\,k/m} \end{aligned}\right.$$

Problem 4.11 instructs the reader to plot these solutions for different combinations of A_+ and A_-.

2. Two complex roots
This discussion is analogous to the one provided in Section 4.5.2. One obtains

$$x(t) = A\, \mathrm{e}^{-\gamma\,(t-t_0)/2}\, \cos\big(\varphi + \omega_\gamma\,(t - t_0)\big) \tag{4.5.5}$$

where A and φ must be fixed based on the initial conditions

$$\begin{aligned} x(t_0) &= A\, \cos(\varphi) \\ \dot{x}(t_0) &= -\omega_\gamma\, A\, \sin(\varphi) \end{aligned}$$

or $$A^2 = z^2(t_0) + \frac{\dot{z}^2(t_0)}{\omega_\gamma^2} \qquad \text{and} \qquad \varphi = \arcsin\left(\frac{\dot{z}(t_0)}{\omega_\gamma\, A}\right)$$

In Problem 4.12 the reader is advised to fill in the details of this derivation.

3. A single double root
For $\gamma^2 = 4\,k/m$ the characteristic polynomial has a single root $\lambda = -\gamma/2$ such that we only find a single solution $\exp(\lambda t)$ of the ODE. The ODE is solved then as follows:

Algorithm 4.4: Linear 2nd order ODEs: the degenerate case

A 2^{nd}-order linear homogeneous ODE whose characteristic polynomial has a double root at $\lambda = c$ takes the form

$$0 = \ddot{h}(t) - 2\,c\,\dot{h}(t) + c^2\,h(t) \quad \text{with} \quad c \in \mathbb{C}$$

This ODE has two independent solutions $\exp(\lambda\, t)$ and $t\, \exp(\lambda\, t)$ such that its general solutions can be written as

$$h(t) = \big(A + B\,(t - t_0)\big)\, \mathrm{e}^{c\,(t-t_0)}$$

Here the amplitudes A and B must be chosen such that the solution obeys the initial condition

$$\left.\begin{aligned} h(t_0) &= h_0 = A \\ \dot{h}(t_0) &= v_0 = c\,A + B \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} A &= h_0 \\ B &= v_0 - c\,h_0 \end{aligned}\right.$$

Remark 4.11. The function $t\,\exp(ct)$ is a solution of the ODE iff the characteristic polynomial has a double root:

Proof.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t\,\mathrm{e}^{c\,t}\right) + a\,\frac{\mathrm{d}}{\mathrm{d}t}\left(t\,\mathrm{e}^{c\,t}\right) + b\,\left(t\,\mathrm{e}^{c\,t}\right) \\
&= \mathrm{e}^{c\,t}\,\left[(2\,c+a) + \left(c^2 + a\,c + b\right)\,t\right] \\
\Rightarrow \qquad & (2\,c+a) = 0 \quad \wedge \quad \left(c^2 + a\,c + b\right) = 0
\end{aligned}
$$

The first equations holds iff $a = -2\,c$ and the second condition implies then that $b = c^2$. □

For the damped harmonic oscillator we have $c = -\gamma/2$ such that

$$
x(t) = \left[x_0 + \left(v_0 + \frac{\gamma\,x_0}{2}\right)\,(t-t_0)\right]\;\mathrm{e}^{-\gamma\,t/2}
$$

is the solution with $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$.

4.5.4 Self Test

Problem 4.10. **Alternative solution for the mass suspended from a spring**

In Equation (4.5.3) we provided the solution of the EOM (4.5.2) of a mass suspended from a spring. Occasionally one also finds the solution given in the form

$$
z(t) = -\frac{mg}{k} + A_1\;\cos\bigl(\omega\,(t-t_0)\bigr) + A_2\;\cos\bigl(\omega\,(t-t_0)\bigr)
$$

What is the relation between these two solutions? How does one find one from the other?

Hint: Start from Equation (4.5.3) and use trigonometric relations.

Problem 4.11. **Overdamped solutions of the damped harmonic oscillator: time dependence and in phase-space portrait**

In this exercise we discuss the form of the overdamped solutions of the damped harmonic oscillator.

a) Consider first ICs where A_+ and A_- are positive. Verify that there is a time t_c where the two contributions to $x(t)$ are equal. Plot $x(t)\,\exp(-\lambda_+)\,(t_c-t_0)/A_+$ as function of $t-t_0-t_c$, choosing a log-scale for the ordinate axis. You should observe linear behavior for large negative and positive values on the mantissa $t-t_0-t_c$. What are the slopes of these lines? What is the value where the function intersects with $t-t_0-t_c=0$?

b) Consider first ICs where $A_+ > 0 > A_-$ and plot $x(t)$ as function of $t-t_0$. Add the functions that describe the asymptotics of $x(t)$ for very small and for large times. You will find that this function has a root and a maximum. Find the time where this happens, and the function value at the maximum.

c) Sketch the motion in phase space! Make use to this end of the special points that you evaluated in b).

Problem 4.12. **Damped oscillations of the damped harmonic oscillator: derivation and phase-space portrait**

For $\gamma^2 > 4\,k/m$ the damped harmonic oscillator shows damped oscillations as given in Equation (4.5.5).

a) Why should the amplitudes $A_\pm$ of the two solutions be complex conjugate?

b) Choose the ansatz $A_\pm = A\,\exp(\pm \mathrm{i}\varphi)/2$ and derive the result provided in Equation (4.5.5).

c) Note that there is no explicit γ dependence of IC.
Why does it drop out?

d) How does this motion look like in phase space?

Problem 4.13. **The solutions of a homogeneous linear ODE form a vector space**

The set of solutions S_N of a homogeneous N^{th}-order homogeneous linear ODE,

$$0 = \sum_{\nu=0}^{N} c_\nu(t)\, f^{(\nu)}(t)\,, \tag{4.5.6}$$

forms a vector space (cf. the Definition 2.8). Proof to this this end that

a) $(\mathsf{S}, +)$ is a commutative group. The non-trivial statement that must be checked to this end is that

$$\forall s_1(t), s_2(t) \in \mathsf{S} : \quad s_1(t) + s_2(t) \in \mathsf{S}$$

b) Verify that

$$\forall \alpha \in \mathbb{C}\,,\ \ s(t) \in \mathsf{S} : \quad \alpha\, s(t) \in \mathsf{S}$$

and show that the other properties of a vector space follow trivially from the properties of real functions.

c) Show that the vector space S_N has dimension N.

d) Show that the functions $\exp(\lambda t)$ are a solution of Equation (4.5.6) iff λ is a root of the characteristic polynomial.

e) In Algorithm 4.3 we wrote the solutions as $h(t) = \sum_k A_k\, \exp(\lambda_k\, t)$.
Show that this can be interpreted as a representation of the vector $h(t)$ as a linear combination with coordinates A_k with respect to a basis $\{\exp(\lambda_k\, t), k = 1, \ldots, N\}$.
Why is it important to this end that the characteristic polynomial has distinct roots?

f) What about inhomogeneous, linear ODEs? Do their solutions form a vector space, too? If yes: proof it! If no: provide counterexamples for all properties that are violated.

add: variation of constants: fish pond, bells

4.6 Motion of interacting particles and celestial mechanics

One of the most important objectives of physics is the description of the motion of interacting particles. As a first step in this direction we discuss the motion of point particles that interact with a conservative force that depends only on the scalar distance between the particles, the interaction most commonly encountered in physical systems. The impact of spatial extension will be the topic of Chapter 5.

Definition 4.7: Point Particles

A *point particle* is an idealization of a physical object where its mass is considered to be concentrated in a single point in space $\boldsymbol{x}$. Point particles can not collide. However, their motion can be subjected to forces that depend on their position $\boldsymbol{x}$.

Example 4.8: Kepler Problem

The Kepler problem addresses the motion of a planet of mass m that orbits around a sun of mass M. The sun and the planet are so far apart that it is justified to consider their masses as concentrated in the positions $\boldsymbol{q}_P$ and $\boldsymbol{q}_S$, and to approximate their interaction as arising from the potential

$$\Phi(R) = \frac{mMG}{R}$$

where $G = 6.672\,59 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is the constant of gravitation and $R = |\boldsymbol{q}_P - \boldsymbol{q}_S|$ is the distance between planet and sun. Planet and sun are considered as point particles.

Remark 4.12. The approximation of point particles has been introduced by Newton upon providing the first mathematical model for the Kepler problem. Subsequently, it has extremely successfully been applied in celestial mechanics. *Celestial Mechanics* addresses the problem of discussing the motion of all planets and their moons based on pair interactions deriving from the potential provided in Example 4.8. How to the tiny interactions between the planets impact their motion over long times? Is our solar system stable, or will–at some time in the far future–some planet or moon borrow energy from the other bodies and escape into outer space? 🕮

Remark 4.13. A straightforward application of the Kepler problem is the discussion of the motion of the Moon around Earth where the predictions have been tested extremely accurately based on satellite data and the return time of light signals send to Moon and reflected by mirrors on its surface that have been left there by space missions. The measurements clearly reveal the limitations of the model: Most noticeably, the Moon gives rise to tidal forces on Earth that induce a tiny amount of dissipation. Even in celestial mechanics there are small dissipative corrections to conservative interaction.

add references and exercises

In Section 3.4 we learned that conservation laws impose constraints on the motion of bodies that can be used to simplify the description of their motion. We consider the motion of N particles of masses m_i, $i = 1, \dots, N$ at the positions $\boldsymbol{q}_i$, $i = 1, \dots, N$ that are subjected to forces $\boldsymbol{F}_{ij}$ acting between every pair (i, j) of particles. There is not self-interaction $\boldsymbol{F}_{ii} = \boldsymbol{0}$, and the forces obey Newtons 3rd law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moreover, they are conservative, and depend only on the distance of the particles, $\boldsymbol{F}_{ij} = \nabla \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|)$. Here the indices ij indicate that the force may depend on additional scalar parameters such as the mass or charge of the particles.

4.6.1 *Center of mass motion and relative motion*

We first determine the evolution of the position of the center of mass $\boldsymbol{Q}$ of the system

$$\boldsymbol{Q} = \frac{1}{M} \sum_i m_i \, \boldsymbol{q}_i \quad \text{with total mass} \quad M = \sum_i m_i \tag{4.6.1}$$

Its evolution is not subjected to external forces

$$\ddot{\boldsymbol{Q}} = \frac{1}{M} \sum_i m_i \, \ddot{\boldsymbol{q}}_i = \frac{1}{M} \sum_i \sum_j \boldsymbol{F}_{ij} = \boldsymbol{0} \tag{4.6.2}$$

due to Newtons 3rd law.

Hence, we find for an initial position $\boldsymbol{Q}_0$ and initial velocity $\boldsymbol{V}_0$ at an initial time t_0 that

$$\boldsymbol{Q}(t) = \boldsymbol{Q}_0 + \boldsymbol{V}_0 \, (t - t_0) \tag{4.6.3}$$

Now we introduce the coordinates relative to the center of mass $\boldsymbol{r}_i = \boldsymbol{q}_i - \boldsymbol{Q}$ and we observe that

$$\begin{aligned}
m_i \, \ddot{\boldsymbol{r}}_i &= m_i \, \ddot{\boldsymbol{q}}_i - m_i \, \ddot{\boldsymbol{Q}} = m_i \, \ddot{\boldsymbol{q}}_i \\
&= \sum_j \boldsymbol{F}_{ij} = -\nabla_{\boldsymbol{q}_i} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) = -\frac{\boldsymbol{q}_i - \boldsymbol{q}_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|} \, \Phi'_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) \\
&= -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \, \Phi'_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|)
\end{aligned} \tag{4.6.4}$$

where $\Phi'_{ij}(x)$ denotes the derivative of $\Phi_{ij}(x)$ with respect to its scalar argument x.

Hence, the EOMs for $\boldsymbol{Q}$ and for the positions $\boldsymbol{r}_i$ relative to the center of mass can be solved separately of each other, and the EOM for the CM has a trivial solution, Equation (4.6.3). We may therefore always address the motion of the particles in a setting where their center of mass is fixed at the origin of the coordinate system.

4.6.2 *Angular momentum in celestial mechanics*

The total angular momentum is conserved for systems where all forces are due to pairwise interactions between particles pairs of particles ij that obey Newtons 3rd law $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$ with forces acting along the line connecting particle i and j, i. e. in particular for the forces of the form given in Equation (4.6.4). After all,

$$\begin{aligned}
\boldsymbol{L} &= \sum_i \boldsymbol{q}_i \times m_i\, \dot{\boldsymbol{q}}_i = \sum_i (\boldsymbol{Q} + \boldsymbol{r}_i) \times m_i\, (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}}_i) \\
&= \sum_i (m_i\, \boldsymbol{Q} \times \boldsymbol{Q} + \boldsymbol{Q} \times m_i\, \dot{\boldsymbol{r}}_i + m_i\, \boldsymbol{r}_i \times \dot{\boldsymbol{Q}} + \boldsymbol{r}_i \times m_i\, \dot{\boldsymbol{r}}_i) \\
&= M\boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times m_i\, \dot{\boldsymbol{r}}_i
\end{aligned}$$

where the terms that contain only a single factor of $\boldsymbol{r}_i$ or $\dot{\boldsymbol{r}}_i$ vanish because $\sum_i m_i\, \boldsymbol{r}_i = \sum_i m_i\, (\boldsymbol{q}_i - \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{Q} = \boldsymbol{0}$. Now we have

$$M\, \ddot{\boldsymbol{Q}} = \sum_i m_i\, \ddot{\boldsymbol{q}}_i = \sum_i \sum_j \boldsymbol{F}_{ij} = \sum_{i<j} \boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = \boldsymbol{0}$$

such that we obtain for the time derivative

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} &= M\, \boldsymbol{Q} \times \ddot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times m_i\, \ddot{\boldsymbol{r}}_i = \boldsymbol{Q} \times M\, \ddot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times \sum_j \boldsymbol{F}_{ij} \\
&= \frac{1}{2} \left(\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} \right) \\
&= \frac{1}{2} \left(\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \sum_{ij} \boldsymbol{r}_j \times \boldsymbol{F}_{ij} \right) \\
&= \frac{1}{2} \sum_{ij} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = \boldsymbol{0}
\end{aligned}$$

Upon moving to the second line we used that $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$, and the antisymmetry of the forces $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moving to the third line we swapped the names of the summation indices i and j. In the last line, we collected terms and used that $\boldsymbol{F}_{ij}$ is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$. We summarize this important finding the following

Theorem 4.1: Angular momentum conservation

The relative angular momentum is conserved for systems with pairwise interaction forces acting parallel to the distance between particles. The total angular momentum is conserved when external forces vanish or when they give rise to a center-of-mass forces $M\ddot{\boldsymbol{Q}}$ aligned parallel to $\boldsymbol{Q}$.

Remark 4.14. An example for the latter case is a harmonic force $\boldsymbol{F}_i = c\, m_i\, \boldsymbol{q}_i$. The proof is provided as Problem 4.17c).

Conservation of the relative angular momentum implies important constraints on the motion. In celestial mechanics this is vividly displayed in the shape of galaxies, solar systems and planetary ring structures. All these systems emerge by the gravitational collapse of large stellar dust clouds. Let cloud be spherically symmetric and uniform initially, consisting of a huge number of small dust particles. By statistical fluctuations the cloud will have an angular momentum, of the order of $M D^2 \omega$ where M is the total mass of the cloud, D is the diameter of the cloud and ω is a tiny number with the unit of a rotation frequency. For a solar system the cloud will collapse until virtually all of its mass is concentrated eventually in the sun in its very center. This involves a change of the diameter of the region holding the mass of about 10^4. For conserved angular momentum the frequency ω is growing by a factor of 10^8. In Problem 4.15 you will show that the initial angular momentum can not be coped by a spin of the central star. The competing constraints of the tendency of gravity to lump together the matter in the cloud and the need to conserve angular moment eventually form a solar system with a central very massive star or double star that is surrounded by planets moving around the star at a distance large as compare to the size of the star.

4.6.3 *Self Test*

Problem 4.14. **The CM of the solar system and the position of the sun**

Verify that the center of mass of Sun can lie more than a sun-diameter away from the center of mass of the solar system.

Hint: Relevant parameters are provided in Table A.1.

Problem 4.15. **Angular momentum of the solar system**

The solar system has a total angular momentum of about $L_{SoSy} = 3.3212 \times 10^{45}\,\mathrm{kgm^2s^{-1}}$.

a) Assume that the mass was initially distributed in a ball of a radius of about 40 AU. Estimate the corresponding effective frequency ω.

b) Assume that the mass in concentrated in two point particles that circulate around each other at a distance of about the sun diameter. Compare their rotation speed to the speed of light.

c) Verify that 98% of L_{SoSy} is accounted for by the orbital angular momenta of the planets.

d) How does this imply the disk-like structure of our solar system?

e) Speculate about other effects that contribute to the remaining 2% of the total angular momentum.

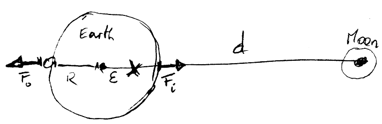


Figure 4.10: Distances adapted for the estimate of the forces inducing tidal forces F_o and F_i: The distance d between Earth and Moon, the Earth radius R, and the distance ϵ between the center of mass of Earth and the joint center of mass of Earth and Moon (indicated by a cross $\times$). Tides emerge at the side facing the Moon (inwards) and opposing the Moon (outwards).

Problem 4.16. **Tidal forces**

Gravitational forces of Moon and centripetal forces due to the rotation with frequency $\Omega = 1/\text{month}$ of Earth around the common center of mass of Earth and Moon (cross in Figure 4.10) give rise to tides. On the outwards facing side the resulting acceleration on a mass element on the Earth surface can be estimated as

$$a_0 = g - (\epsilon + R)\,\Omega^2 + \frac{G\,M_M}{(d+R)^2}$$

where M_M is the Moon mass.

a) Assume that Earth and Moon evolve on circular paths and employ the force balance for a stable motion in order to show that

$$a_0 = g - R\,\Omega^2 \left[1 - \frac{M_M}{M_E + M_M}\,\frac{R}{d}\,\left(1 + \mathcal{O}(R/d)\right)\right]$$

and determine the higher-order correction terms that are indicated here as $(1 + \mathcal{O}(R/d))$.

b) Determine also the change of the acceleration on the side towards the moon. How does it differ from a_i?

c) Determine the relative change of the gravitational acceleration due to the presence of moon, and the difference between a_i and a_o.

d) So far we only discussed the component of the acceleration along a line connecting Earth and Moon at the innermost and outermost points of the Earth surface. What about the other components of the gravitational acceleration:
when considering tides at mid latitudes?
at positions half-way between the two points (i. e. top and bottom sides of Earth in the figure).

e) What is the impact of the Earth rotation? How does it break the symmetry? What does this imply about the relative strength of the two tidal waves every day?

Problem 4.17. **Center of mass and constants of motion**
How do the expressions for the constants of motion discussed in Section 3.4 behave when separating the center of mass motion and the relative motion, $\boldsymbol{q}_i(t) = \boldsymbol{Q}(t) + \boldsymbol{r}_i(t)$.

a) Show that the kinetic energy $T = \sum_i m_i \dot{\boldsymbol{q}}_i^2$ takes the new value

$$T = \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 + \sum_i \frac{m_i}{2}\,\dot{\boldsymbol{r}}_i^2$$

b) Assume that the system is moving in a gravitational field, and that the other forces on the particle arise from pair-wise conservative interactions as discussed Equation (4.6.4). Show that the total energy can be written as

$$E = \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 - M\boldsymbol{g}\cdot\boldsymbol{Q} + \sum_i \frac{m_i}{2}\,\dot{\boldsymbol{r}}_i^2 + \sum_{i<j} \Phi_{ij}\left(|\boldsymbol{r}_i - \boldsymbol{r}_j|\right)$$

c) Show that the total angular momentum is conserved for a systems with the particles interactions given in Equation (4.6.4) and an additional external force

$$\boldsymbol{F}_i = c\,m_i\,\boldsymbol{q}_i$$

acting on each particle i.

4.7 Motion of two particles and the Kepler problem

The first problem tackled in theoretical mechanics was the motion of two point particles with gravitational interaction. It is formulated in terms of three laws. The second law holds for all central forces, the 3rd law is a consequence of mechanical similarity, and the 1st law is based on a solution of the EOM. We first explore the general arguments, and then illustrate their application to the Kepler problem.

4.7.1 Conservation of angular momentum and Kepler's 2nd Law

Angular momentum conservation also has important consequences for the motion of two particles. The center of mass of the two particles takes the form

$$\boldsymbol{Q} = \frac{m_1}{m_1+m_2}\,(\boldsymbol{Q}+\boldsymbol{r}_1) + \frac{m_2}{m_1+m_2}\,(\boldsymbol{Q}+\boldsymbol{r}_2) = \boldsymbol{Q} + \frac{m_1\,\boldsymbol{r}_1 + m_2\,\boldsymbol{r}_2}{m_1+m_2}$$

such that

$$m_1\,\boldsymbol{r}_1 + m_2\,\boldsymbol{r}_2 = \boldsymbol{0} \quad \text{and in particular} \quad \boldsymbol{p} = m_2\,\dot{\boldsymbol{r}}_2 = -m_1\,\dot{\boldsymbol{r}}_1\,.$$

This has important consequences for the evolution of the conserved angular momentum of the relative motion

$$\boldsymbol{L} = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{p}\,.$$

In view of

$$\begin{aligned}\boldsymbol{p} = m_2\,\dot{\boldsymbol{r}}_2 = m_2\,\dot{\boldsymbol{q}}_2 - m_2\,\dot{\boldsymbol{Q}} &= \frac{m_1 m_2}{m_1+m_2}\,\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q}_2-\boldsymbol{q}_1)\\ &= \mu\,\dot{\boldsymbol{R}} \quad \text{with} \quad \mu = \frac{m_1 m_2}{m_1+m_2} \quad \text{and} \quad \boldsymbol{R} = \boldsymbol{q}_2 - \boldsymbol{q}_1\end{aligned}$$

the angular momentum of the relative motion can be expressed in terms of the vector $\boldsymbol{R}$ connecting the two masses

$$\boldsymbol{L} = \boldsymbol{R} \times \mu\dot{\boldsymbol{R}}$$

It takes the then form the angular momentum of a single particle with mass μ, and this also applies for the relation between the acceleration and the force

$$\mu\ddot{\boldsymbol{R}} = m_2\,\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}\,.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} \times \mu\dot{\boldsymbol{R}} = \boldsymbol{R} \times \boldsymbol{F} = \boldsymbol{0}$$

for forces $\boldsymbol{F}$ acting along the line $\boldsymbol{R}$ connecting the two particles.

The conservation of angular momentum has two important consequences:

1. The direction of $\boldsymbol{L}$ is fixed. As a consequence the positions and the velocities of the planet and the sun always lie in a plane that is orthogonal to $\boldsymbol{L}$, and the force $\mu\ddot{\boldsymbol{R}}$ also lies in the plane because the

force is parallel to $\boldsymbol{R}$. Therefore, the motion is constrained to the plane for all times.

2. The absolute value of $\boldsymbol{L}$ is fixed, and this has a geometric interpretation that was first formulated in the context of planetary motion

Theorem 4.2: Kepler's second law

A segment joining the two particles, planet and sun in the Kepler problem, sweeps out equal areas Δa in equal time intervals Δt.

Proof. For the time interval $[t_0, t_1]$ with length $\Delta t = t_1 - t_0$ one has

$$|\boldsymbol{L}|\,\Delta t = \int_{t_0}^{t_1} \mathrm{d}t\,|\boldsymbol{R} \times (m_2 \boldsymbol{v}_2)| = m_2 \int_{t_0}^{t_1} \mathrm{d}t\,|\boldsymbol{R}|\,|\boldsymbol{v}_2| \sin\alpha$$

where α is the angle between $\boldsymbol{R}$ and $\boldsymbol{v}_2$. Further, $\mathrm{d}\boldsymbol{s} = \boldsymbol{v}_2\,\mathrm{d}t$ is the path length that the trajectory traverses in a time unit $\mathrm{d}t$, such that $\mathrm{d}a = \mathrm{d}t\,|\boldsymbol{R}|\,|\boldsymbol{v}_2| \sin\alpha/2$ is the area swiped over in $\mathrm{d}t$ (see the sketch in Figure 4.11). Hence,

$$|\boldsymbol{L}|\,\Delta t = \frac{1}{2}\int_0^{\Delta a} \mathrm{d}a = \Delta a \qquad \Rightarrow \qquad \Delta a = \frac{2\,|\boldsymbol{L}|}{m_2}\,\Delta t$$

such that Δa is proportional to Δt. □

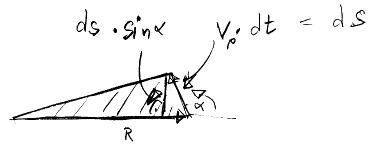


Figure 4.11: Area passed over by the trajectory.

4.8 Mechanical Similarity and Kepler's 3rd Law

Two solutions of a differential equations are called *similar* when they can be transformed into one another by a rescaling of the time-, length-, and mass-scales. We indicate the rescaled quantities by a prime, and denote the scale factors as τ, λ, and α, respectively,

$$t' = \tau t\,, \qquad q_i' = \lambda \boldsymbol{q}_i\,, \qquad m_i' = \alpha m_i$$

We explore the consequences of this idea for the Kepler problem, i.e. for two point particles interacting by a gravitation force $\boldsymbol{F}$ deriving from the following potential

$$\Phi(|\boldsymbol{R}|) = \frac{m_1 m_2 G}{|\boldsymbol{R}|} \qquad \Rightarrow \qquad \boldsymbol{F} = -\nabla\Phi(|\boldsymbol{R}|) = \frac{m_1 m_2 G}{|\boldsymbol{R}|^3}\,\boldsymbol{R}$$

The setup for a planet going around the sun is sketched in Figure 4.12. acting on the planet and pointing towards the sun. We only consider the relative motion and assume that there are no other forces acting on the sun and the planet.

Information about the period and the shape of the trajectory is obtained from the energy for the relative motion

$$E = \frac{\mu}{2}\,\dot{\boldsymbol{R}}^2 + \Phi(|\boldsymbol{R}|)$$

This energy is conserved because

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\,\dot{\boldsymbol{R}}^2 + \Phi(|\boldsymbol{R}|)\right) = \mu\,\dot{\boldsymbol{R}}\cdot\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}\cdot\nabla\Phi(|\boldsymbol{R}|)$$
$$= \dot{\boldsymbol{R}}\cdot\left(\mu\,\ddot{\boldsymbol{R}} - \boldsymbol{F}\right) = \boldsymbol{0}$$

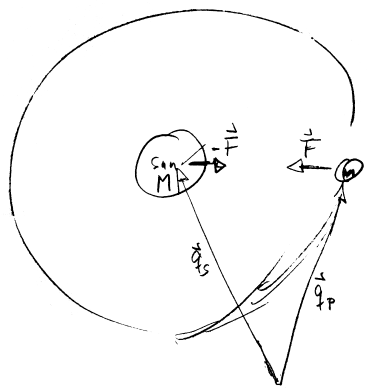


Figure 4.12: Setup of and notations for the motion of a planet around the sun. Here m_S and m_P are the mass of the sun and the planet, respectively, and $\boldsymbol{q}_S$ and $\boldsymbol{q}_P$ are their positions. The relative position is $\boldsymbol{R} = \boldsymbol{q}_P - \boldsymbol{q}_S$.

adapt Figure 4.12

Here, we used that $\boldsymbol{F} = \mu\, \ddot{\boldsymbol{R}}$.

In our planetary system the trajectories of the planets are all circular to a good approximation. They are therefore described by the *same* solution of the EOM up to a rescaling of the length scale and the time scale. The former accounts to their different distance to the Sun, and the latter to the different periods of their motion. We observe now that $\mu = m_1\, m_2/(m_1 + m_2)$ and that the Sun mass m_S is 1000 times larger than the mass of Jupiter, the largest planet. Therefore, for the motion of the planets we have $(m_1 + m_2)/\mu \simeq m_S$ and

$$\frac{E}{\mu} \simeq \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{m_S\, G}{|R|}$$

We expect that different planets follow the same trajectory up to rescaling space and time units, and a different constant value of their energy. We hence explore the consequences of the scaling $\lambda \boldsymbol{R}(t)$ and τt

$$\begin{aligned} \frac{E}{\mu} &\simeq \frac{\lambda^2}{\tau^2}\frac{\dot{\boldsymbol{R}}^2}{2} + \frac{1}{\lambda}\frac{m_S\, G}{|R|} \\ \Leftrightarrow \frac{E\, \tau^2}{\mu\, \lambda^2} &\simeq \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{\tau^2}{\lambda^3}\frac{m_S\, G}{|R|} \end{aligned}$$

where the right-hand side remains invariant iff $\tau^2/\lambda^3 = \text{const}$. This entails

Theorem 4.3: Kepler's third law

The square of the period T of the planets in our planetary system are proportional to the third power of their distance D to the sun.

4.9 *Solving ODEs by coordinate transformations: Kepler's 1st law*

In polar coordinates $\boldsymbol{R} = (R, \theta)$ the kinetic energy takes the form $\mu \dot{\boldsymbol{R}}^2/2 = \mu\left(\dot{R}^2 + (R\dot{\theta})^2\right)/2$ while the conservation of angular momentum implies $R\dot{\theta} = L/(\mu R)$ with $L = |\boldsymbol{L}|$. Consequently,

$$E = \frac{\mu}{2}\, \dot{R}^2(t) + \frac{L^2}{2\mu\, R^2(t)} - \frac{m_1 m_2 G}{R(t)} \tag{4.9.1}$$

which is equivalent to the motion of a particle of mass μ at position R in the one-dimensional effective potential

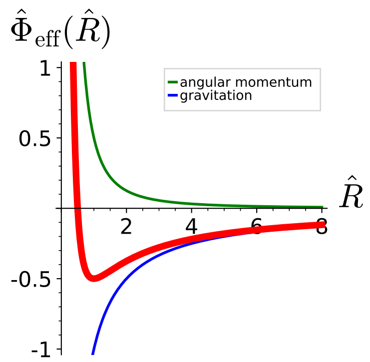


Figure 4.13: Effective potential $\hat{\Phi} = \Phi\, R_0/m_1\, m_2\, G$ for the Kepler problem as function of the dimensionless distance $\hat{R} = R/R_0$, where $R_0 = L^2/\mu\, m_1\, m_2\, G$.

$$\Phi_{\text{eff}}(R) = \frac{L^2}{2\mu\, R^2} - \frac{m_1 m_2 G}{R}\,.$$

The first, repulsive contribution to the effective potential arises from angular momentum conservation, and the second, attractive contribution is due to gravity.

There is no elementary way to determine the function $R(t)$. However, based on Equation (4.9.1) one can plot the trajectories in phase space, $\dot{R}(R)$ for different energies. This plot is provided in Figure 4.14. For negative energies there are bounded trajectories that oscillate in the minimum of the potential Φ_{eff}. For zero energy the trajectory reaches till $R = \infty$, and reaches infinity with zero speed. For a positive energy the trajectory reaches till $R = \infty$, and it will go there with speed $\dot{R} = \sqrt{2E/\mu}$.

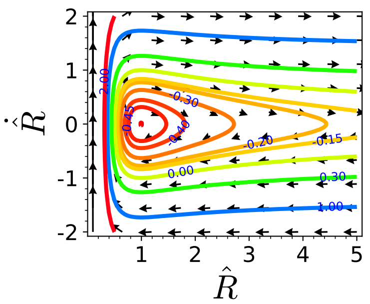


Figure 4.14: The phase-space flow for the EOM of $R(t)$ provided by Equation (4.9.1). The plot adopts dimensionless units with length scale R_0 introduced in Figure 4.13 and a time scale $t_0 = \sqrt{\mu\, R_0^3 / m_1\, m_2\, G}$. Solid lines refer to solutions for different dimensionless energy, with values marked on the contour lines.

However, one can determine the shape $R(\theta)$ of the trajectories by observing

$$\dot{R}(\theta) = \dot{\theta}\, \frac{\mathrm{d}R(\theta)}{\mathrm{d}\theta} = \frac{L}{\mu\, R^2}\, R'(\theta)$$

such that

$$E = \frac{L^2}{2\,\mu} \left(\frac{R'^2}{R^4} + \frac{1}{R^2} \right) - \frac{m_1\, m_2\, G}{R}$$

In terms of $w(\theta) = 1/R(\theta)$ this implies

$$\frac{\mu\, E}{L^2} = \frac{1}{2}\, \big(w'(\theta)\big)^2 + \frac{1}{2}\, w^2(\theta) - \frac{m_1\, m_2\, \mu\, G}{L^2}\, w(\theta) \tag{4.9.2}$$

and differentiating with respect to θ provides

$$0 = w'(\theta)\, \left[w''(\theta) + w(\theta) - \frac{m_1\, m_2\, \mu\, G}{L^2} \right] .$$

The expression in the square bracket is a second order linear ODE with solution

$$w(\theta) = \frac{\mu\, m_1\, m_2\, G}{L^2}\, \left[1 + \epsilon\, \cos\left(\theta - \theta_0\right)\right]$$

where ϵ and θ_0 are integration constants that must be determined from the initial conditions. Inserting $w(\theta)$ into Equation (4.9.2) yields

$$\frac{\mu\, E}{L^2} = \frac{\epsilon^2 - 1}{2}\, \left(\frac{m_1\, m_2\, \mu\, G}{L^2} \right)^2 \quad \Rightarrow \quad \epsilon^2 = 1 + \frac{2\, E\, L^2}{\mu\, \left(m_1\, m_2\, G\right)^2}$$

Hence, ϵ is fully determined by the parameters and the conservation laws of the Kepler problem, while θ_0 determines the orientation of the trajectory in the plane. Commonly, one chooses the coordinate frame where $\theta_0 = 0$.

For the motion of a planet around the sun this entails

Theorem 4.4: Kepler's first law

The trajectories of planets around the sun are described by sections of the cone with a plane,

$$R(\theta) = \frac{R_0}{1 + \epsilon\, \cos(\theta - \theta_0)} \tag{4.9.3}$$

where $R_0 = L^2/m_1\, m_2\, \mu\, G$ sets the length scale of the trajectory and θ_0 the orientation in the plane. The parameter ϵ sets its shape: for $\epsilon = 0$ the shape amounts to a circle with radius R_0, for $0 < \epsilon < 1$ to an ellipse, for $\epsilon = 1$ to a parabola, and for $\epsilon > 1$ to a hyperbola.

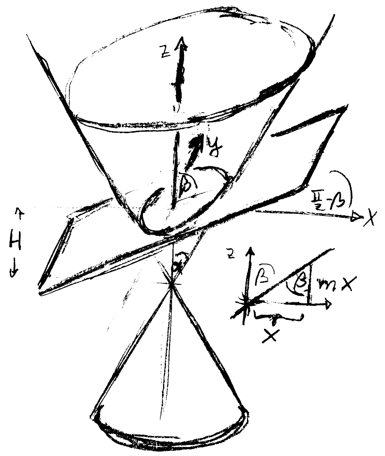


Figure 4.15: Section of a cone double and a plane. The axis are drawn here at the intersection point of the plane with the cone axis, in order to emphasize the the plane is tilted around the y axis. For calculations in the main text the vertex of the cone will be chosen as origin of the coordinate system.

Proof. We consider the section of a cone with opening angle α and its symmetry axis aligned along the z-axis, and a plane, as sketched in Figure 4.15. The origin of the coordinate system is a the vertex of the cone. The plane is tilted with respect to the y-axis such that it forms an angle β with the z-axis, and it intersects the z-axis at height H. The points in the plane have coordinates

$$\boldsymbol{q} = \begin{pmatrix} x \\ y \\ H + m\,H \end{pmatrix} \qquad \text{with} \quad m^{-1} = \tan\beta$$

The point $\boldsymbol{q}$ lies on the cone when $\boldsymbol{q}\cdot\hat{\boldsymbol{z}} = |\boldsymbol{q}|\,\cos\alpha$, which entails

$$(H + m\,x)^2 = \cos^2\alpha\,\left(x^2 + y^2 + (H + m\,x)^2\right)$$

Henceforth, we adopt dimensionless coordinates $\hat{x} = x/H\,\tan\alpha$ and $\hat{y} = y/H\tan\alpha$, and we introduce the abbreviation $\epsilon = m\,\tan\alpha$. We will denote the distance from the origin in the (x, y)-plane as $R = \sqrt{\hat{x}^2 + \hat{y}^2}$, and introduce θ such that $\hat{x} = R\,\cos\theta$. This entails

$$(1 + \epsilon\,R\,\cos\theta)^2 = R^2$$

with solutions

$$\begin{aligned} R_{\pm} &= \frac{\epsilon\,\cos\theta}{1 - \epsilon^2\,\cos^2\theta} \pm \frac{\left[\epsilon^2\,\cos^2\theta + \left(1 - \epsilon^2\,\cos^2\theta\right)\right]^{1/2}}{1 - \epsilon^2\,\cos^2\theta} \\ &= \frac{\epsilon\,\cos\theta \pm 1}{1 - \epsilon^2\,\cos^2\theta} = \frac{-1}{\pm 1 + \epsilon\,\cos\theta} \end{aligned}$$

Hence, Equation (4.9.3) describes a cone section with length scale $R_0 = H\,\tan\alpha$ and eccentricity $\epsilon = m\,\tan\alpha$.

The eccentricity amounts to the ratio of the slope m of the z coordinates of points in the plane as function of x, and the slope $1/\tan\alpha$ of the line obtained as intersection of the double cone and the (x, z) plane. This ratio determines the shape of the conic section (see Figure 4.16).

For $\epsilon = 0$ the shape is a circle $R^2 = 1$.

For $\epsilon = 1$ the shape is a parabola described by $1 + 2\,\epsilon\,\hat{x} = \hat{y}^2$.

For $0 < \epsilon < 1$ the shape is an ellipse described by

$$\frac{1+\epsilon^2}{1-\epsilon^2} = \hat{y}^2 + \left(1 - \epsilon^2\right)\,\left(\hat{x} - \frac{\epsilon}{1-\epsilon^2}\right)^2$$

For $1 < \epsilon$ the shape is a hyperbola described by

$$\hat{y} = \pm\sqrt{\frac{-1}{\epsilon^2 - 1} + (\epsilon^2 - 1)\,\left(\hat{x} + \frac{\epsilon}{\epsilon^2 - 1}\right)^2}$$

$\square$

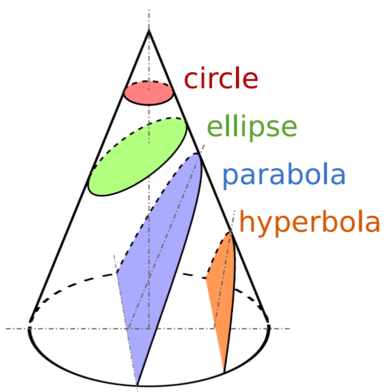


Figure 4.16: Shape of conic sections for parameters, $\epsilon = 0$ circle, $0 < \epsilon < 1$ ellipse, $\epsilon = 1$ parabola, and $\epsilon > 1$ hyperbola.

add remark on physical interpretation of cone and plane

4.9.1 Self Test

Problem 4.18. **Keeping the Moon at a distance**
Something goes wrong at the farewell party for the settlers of the new Moon colony *Sleeping Beauty 1* such that an extremely annoyed evil fairy switches off gravity for the Moon. Luckily there also is a good fairy at the party. She cannot undo the curse but offers to strip all protons from all water-molecules in a bucket of water that you give to her, and hide them on Moon. The Coulomb attraction between electrons on Earth and protons on Moon can then undo the damage.

a) How much water would you give to her?

b) What will happen to the Earth-Moon system when you are off by 20%, by a factor of two, or even by an order of magnitude?
Hint: The idea is that you discuss the motion for an initial condition where Earth and Moon are at their present position and move with their present velocity, while the gravitational force is changed by a the specified factor.

Problem 4.19. **Mechanical similarity and dimensional analysis**
We discuss here the relation between dimensional analysis, introduced in Section 1.2, and mechanical similarity, adopting the notations introduced in the beginning of Section 4.8.

a) We consider a system with kinetic energy $T = \frac{1}{2} \sum_i m_i \dot{\boldsymbol{q}}_i^2$, and consider a potential that admits the following scaling

$$V' = \mu^\alpha \lambda^\beta V$$

Show that the EOM are then invariant when one rescales time as

$$\tau = \mu^{(1-\alpha)/2} \lambda^{(2-\beta)/2}$$

b) Consider now two pendulums, $V = mgz$ with different masses and length of the pendulum arms. Which factors τ, λ, and μ relate their trajectories? How will the periods of the pendulums thus be related to the ratio of the mass and the length of the arms? Which scaling do you expect based on a dimensional analysis?

c) What do you find for the according discussion of the periods of a mass subjected to a harmonic force, $V = k\,|\boldsymbol{q}|^2/2$?

d) Discuss the period of the trajectories in the Kepler problem, $V = mMG/|\boldsymbol{q}|$. In this case the dimensional analysis is tricky because the masses of the sun and of the planet appear in the problem. What does the similarity analysis reveal about the relevance of the mass of the planet for Kepler's third law?

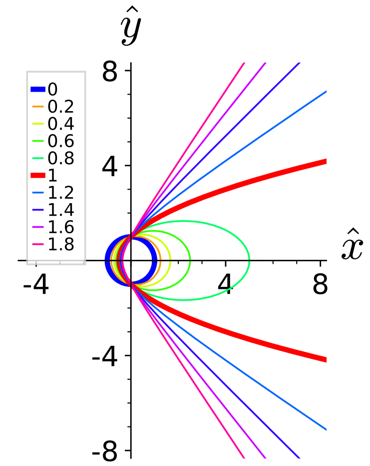


Figure 4.17: Conic sections for different eccentricity ϵ.

Problem 4.20. **Conic Sections**

In the margin we show the shape of conic sections for different eccentricity ϵ.

a) Show that all conic sections intersect the $\hat{y}$ axis at ± 1.

b) Show that the conic sections intersect the $\hat{x}$ axis at $-1/(\epsilon \pm 1)$. Where are these points located for different conic sections?

c) In the proof of Theorem 4.4 we found two different solutions $R_\pm$ for the radius. Which one did you sketch in your plots c)–e)? What about the other solution?

d) How does Equation (4.9.1) look like after introducing the dimensionless units adopted in Figures 4.13 and 4.14? Write down the solution of the EOM in dimensionless units.

e) Find an alternative choice of the length scale such that all trajectories intersect the x-axis at the potion -1, and prepare the corresponding plot of the trajectory shapes in the (x, y)-plane.

4.10 Problems

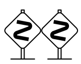
4.10.1 Rehearsing Concepts

Problem 4.21. **Maximum distance of flight.**

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.



What is the optimum choice of θ for the shot put?

c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

Problem 4.22. **Phase-space portraits for a scattering problem**

a) Sketch the potential $\Phi(x) = 1 - 1/\cosh x$ for $x \in \mathbb{R}$.

b) Sketch the direction field in the phase space for the EOM $\ddot{x} = -\partial_x \Phi(x)$.

c) Show that $E = \frac{1}{2}\dot{x}^2 + \Phi(x)$ is a constant of motion of the EOM.

d) Use energy conservation to determine the shape of the trajectories in phase space, and add a few trajectories to the plot started in b).

Add to the sketch a the phase portrait of the motion in this potential, i. e. the solutions of in the phase space $(x, \dot{x})$.

Problem 4.23. **Another linear ODEs with constant coefficients.**

Consider the ODE

$$\ddot{x} = a\,x \qquad \text{with} \quad a \in \mathbb{R}_+$$

a) Sketch the direction field in phase space.

b) Find the solutions of $x(x)$.

c) Add the trajectories the are proceeding through the points $(x(t_0), \dot{x}(t_0)) \in \{(1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1), (1,-1)\}$ to the plot started in a).
Hint: Only two cases must be solved explicitly. All other solutions can be inferred from symmetry arguments.

Problem 4.24. **Stokes drag.**

The EOM for Stokes friction, Equation (4.3.1) is a linear differential equation. Adopt the strategy for solving linear differential equations, Algorithm 4.3, to find the solution Equation (4.3.3b).

4.10.2 *Practicing Concepts*

Problem 4.25. **Egyptian water clocks**

In ancient Egypt time was measured by following how water is running out of a container with a constant cross section A. At a water level h in the container, the water will then run out at a speed

$$v(t) = -c\,\sqrt{2g\,h(t)}$$

where the numerical constant c accounts for the viscosity of water and the geometry of the vessel. The Egyptian water clocks this constant takes values of the order of $c \simeq 0.6$.

a) How does the height $h(t)$ of the water in he container evolve after the plug is pulled?

For use as a clock it would be desirable to change the design of the clock such that $h(t)$ would decrease linearly in time. How can the construction of the water clock be amended to reach that aim?

Problem 4.26. **Damped oscillator.**

Physical systems are subjected to friction. This can be taken into account by augmenting the EOM of a particle suspended from a spring, Equation (4.5.1), by a friction term

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) - \mu\,\dot{z}(t)$$

a) How does friction affect the motion $z(t)$ of the particle? What is the condition that there are still oscillations, even though with a damping? For which parameters will they disappear, and how do the solutions look like in that case?

b) Sketch the evolution of the trajectories in phase space, for the two settings with and without oscillations.

c) For the borderline case the characteristic polynomial will only have a single root, λ. Verify that the general solution can then be written as
$$z(t) = z_0 + A_1\,\mathrm{e}^{\lambda\,(t-t_0)} + A_2\,t\,\mathrm{e}^{\lambda\,(t-t_0)}$$

d) Determine the solutions for a particle for the following initial conditions:
the particle is at rest and at a distance A from its equilibrium position,
the particle is at the equilibrium position, but it has an initial velocity v_0.
Indicate the form of these trajectories in the phase-space plots.

Problem 4.27. **One-dimensional collisions in the center-of-mass frame.**

In Example 3.11 we discussed one-dimensional collisions for settings where the second particle is initially at rest. Now, we consider the situation where both particles are moving from the beginning. Specifically, we consider a setting with two particles of masses m_1 and m_2 with the initial conditions $\big(q_1(t_0), v_1\big)$ and $\big(q_2(t_0), v_2\big)$.

a) Show that the center of mass $Q(t) = \big(m_1\,x_1(t) + m_2\,x_2(t)\big)/M$ with $M = m_1 + m_2$ of the two particles evolves as
$$Q(t) = Q(t_0) + \dot{Q}(t_0)\,(t-t_0) \quad \text{where} \quad \dot{Q}(t_0) = a_1\,v_1 + a_2\,v_2$$
and determine the associated real constants a_1 and a_2.

b) We denote the relative coordinates as $x_i = q_i - Q$ and associate it with a momentum $m_i\,\dot{x}_i$. Show that the relative momenta add up to zero before and after the collision,
$$0 = m_1\,x_1 + m_2\,x_2 = m_1\,(q_1 - Q) + m_2\,(q_2 - Q)$$

and that they swap signs upon collision.
Hint: This is a consequence of energy conservation.

c) Determine the time evolution before and after the collision.

d) Verify the consistency of your result with the special case treated in Example 3.11.

planet swings, lambda points

Problem 4.28. **Motion in a harmonic central force field**

A particle of mass m and at position $\boldsymbol{r}(t)$ is moving under the influence of a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = -k\,\boldsymbol{r}\,.$$

a) We want to use the force to build a particle trap,[2] i.e. to make sure that the particle trajectories $\boldsymbol{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\boldsymbol{r}(t)| < B$ for all times t. What is the requirement on k to achieve this aim?

b) Determine the energy of the particle and show that the energy is conserved.

c) Demonstrate that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m\,\dot{\boldsymbol{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

d) Let (x_1, x_2) be the coordinates in the plane that is singled out by the angular momentum conservation. Show that $m\ddot{x}_i(t) + k\,x_i(t) = 0$ for $i \in \{1, 2\}$. Determine the solution of these equations. Sketch the trajectories in the phase space $(x_i, \dot{x}_i)$. What determines the shape of the trajectories?

e) Show that the trajectories in the configuration space (x_1, x_2) are ellipses. What determines the shape of these trajectories?

f) Discuss the relation between the amplitude and shape of the trajectory, as determined by the ratio and the geometric mean of the major axes of the ellipse in configuration space, and the period of the trajectory.

[2] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

4.10.3 Mathematical Foundation

Problem 4.29. **Differential Equations and functional dependencies**

Determine ODEs whose general solutions are of the form

a) $y(x) = C\,x^2 - x$

b) $y^2(x) = A\,x + B$

Here, A, B, and C are real constants that will be determined by the IC of the ODE.

Problem 4.30. **Separation of variables for a non-autonomous ODE**

We consider the ODE

$$y'(x) = \frac{x}{y}$$

a) How many degrees of freedom does this system have? What is its space? State it as a first order ODE in terms of the phase-space variables.

b) Sketch the direction field in phase space.

c) Find the solution of the ODE for ICs (x_0, y_0) with $y_0 \neq 0$ and

i. $x_0 < 0$ and $x_0 < y_0 < -x_0$

ii. $x_0 > 0$ and $x_0 > y_0 > -x_0$

iii. other ICs with $|x_0| \neq |y_0|$

iv. $|x_0| = |y_0|$

d) Determine the largest interval of values $x \in \mathbb{R}$ where the solutions $y(x)$ obtained in b) are defined.

e) Is the function $y(x) = |x|$ a solution of the ODE? If in doubt: Where do you see problems for this solution?

Problem 4.31. **Effective potentials and phase-space portraits.**

We consider ODEs of the form

$$\ddot{x}(t) = -\frac{\mathrm{d}}{\mathrm{d}x} V_{\text{eff}}(x)$$

Sketch the solutions for trajectories in the following potentials in the phase space $(x, \dot{x})$.

a) $V_{\text{eff}} = x \sin x$ b) $V_{\text{eff}} = x \cos x$
c) $V_{\text{eff}} = x - \sin x$ d) $V_{\text{eff}} = x - \cos x$
e) $V_{\text{eff}} = \mathrm{e}^{x} \sin x$ f) $V_{\text{eff}} = \mathrm{e}^{-x} \sin x$

Problem 4.32. **Central forces conserve angular momentum**

Consider a system of N particles at the positions $\boldsymbol{q}_i$ with masses m_i where each pair (ij) interacts by a force $\boldsymbol{F}_{ij}(|\boldsymbol{d}_{ij}|)$ acting parallel to the displacement vector $\boldsymbol{d}_{ij} = \boldsymbol{q}_j - \boldsymbol{q}_i$ from particle i to j. Proof the following statements:

a) The evolution of the center of mass of the system

$$\boldsymbol{Q} = \frac{1}{M} \sum_{i=0}^{N} m_i \boldsymbol{q}_i \quad \text{with} \quad M = \sum_{i=0}^{N} m_i$$

is force free, i. e. $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$.

b) The total angular momentum can be written as

$$\boldsymbol{L}_{\text{tot}} = M\, \boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \sum_{i<j} \mu_{ij}\, \boldsymbol{d}_{ij} \times \dot{\boldsymbol{d}}_{ij}$$

Determine the factors μ_{ij}.

c) The two contributions to the angular momentum, $M\, \boldsymbol{Q} \times \dot{\boldsymbol{Q}}$ and the sum $\sum_{i<j} \mu_{ij}\, \boldsymbol{d}_{ij} \times \dot{\boldsymbol{d}}_{ij}$ are both conserved.

Problem 4.33. **Impact of translations on conservation laws**

We consider a coordinate transformation where the origin of the coordinate systems is moved to a new time-dependent position $\boldsymbol{x}(t)$,

$$\boldsymbol{q}_i(t) = \boldsymbol{x}(t) + \boldsymbol{r}_i(t)$$

a) Show that the expressions for the kinetic energy are related by

$$T = \sum_i \frac{m_i}{2}\, \dot{\boldsymbol{q}}_i^2 = \frac{M}{2}\, \dot{\boldsymbol{x}}^2 + M\, \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{Q}} + \sum_i \frac{m_i}{2}\, \dot{\boldsymbol{r}}_i^2$$

Here, $M = \sum_i m_i$ and $\boldsymbol{Q} = M^{-1} \sum_i m_i \boldsymbol{q}_i$ are the total mass and the center of mass, respectively.

b) Show that the expressions for the total energy for motion in an external field are related by

$$E = T - M\, \boldsymbol{g} \cdot \boldsymbol{Q} + \sum_{i<j} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) = T - M\, \boldsymbol{g} \cdot \boldsymbol{Q} + \sum_{i<j} \Phi_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) - M$$

c) Show that the angular momentum transforms as follows

$$L = \sum_i m_i\, \boldsymbol{q}_i \times \dot{\boldsymbol{q}}_i = M\, \boldsymbol{x} \times \dot{\boldsymbol{Q}} + M\, (\boldsymbol{x} + \boldsymbol{Q}) \times \dot{\boldsymbol{x}} + \sum_i m_i\, \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i$$

d) Show that conservation laws are mapped to conservation laws iff we consider a Galilei transformation, i. e. a transformation where $\dot{x}$ =const.

4.10.4 *Transfer and Bonus Problems, Riddles*

Problem 4.34. **Light intensity at single-slit diffraction**

Monochromatic light of wave length λ that is passion through a slit will produce an diffraction pattern on a screen where the intensity follows (cf. Figure 4.18, top panel)

$$I(x) = I_{\text{max}} \left(\frac{\sin x}{x} \right)^2$$

Here the light intensity $I(x)$ is the power per unit area that is observed at a distance x to the side from the direction straight ahead from the light source through the slit to the screen. We are interested in the total power $P(\Delta)$ that falls into a region of width $|x| < \Delta$. Since there is no antiderivative for $I(x)$ we will find approximate solutions by considering Taylor approximations of $I(x)$ that can be integrated without effort.

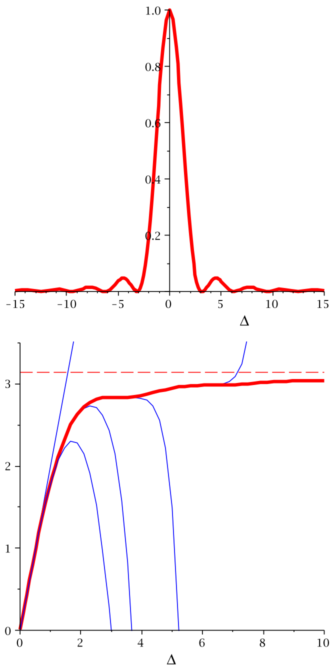


Figure 4.18: The upper panel shows the light intensity $I(x)/I_{\text{max}}$, and the lower panel the fraction of light in the center region of width Δ, i. e. the power $P(\Delta) = \left[\int_{-\Delta}^{\Delta} I(x)\,\mathrm{d}x \right] / I_{\text{max}}$. The red dotted value marks the asymptotic value π and the blue line the approximations obtained by a Taylor approximation up to order 2, 4, 8, 16, and 32. according to the Taylor series evaluated in Problem 4.34.b).

a) Show that $\sin^2 x = (1 - \cos 2x)/2$, and use the Taylor expansion of the cosine-function to show that

$$\frac{\sin^2 x}{x^2} = \frac{1-\cos 2x}{2x^2} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!}\,(2x)^{2n}$$

b) Determine the Taylor approximations for $P(\Delta)$ by integrating the expression found in a).

c) Write a program that is numerically determines $P(\Delta)$ and compares it to Taylor approximations of different order, as shown in the lower panel of Figure 4.18.

Problem 4.35. **Tricky issues in a classical population model**
The Lotka-Volterra model is considered the first model addressing the evolution of populations in theoretical biology. It predicts oscillations of populations, and still today it is cited in the context of data of Lynx and Hare that were collected in Canada in the late 19th century (cf. Figure 4.19).

Let $H(t)$ be the population of prey animals (Hare) and $L(t)$ be the population of its predator (Lynx). When there are no predators the population of prey grows exponentially with a rate a, and this rate is reduced by $-bL(t)$, when prey is consumed by predators. In absence of food the predators die at a rate d, and this rate is reduced by $-cH(t)$, when they find food.

$$\begin{aligned}\dot{H}(t) &= H(t)\;\;[a - b\,L(t)]\\ \dot{L}(t) &= L(t)\;\;[c\,H(t) - d]\end{aligned}$$

a) Let $u(\tau) \propto H(t)$, $v(\tau) \propto L(t)$, and $\tau \propto t$. Find suitable proportionality constants and a dimensionless parameter Π such that

$$\begin{aligned}\dot{u}(\tau) &= \quad u(\tau)\;\;[1 - v(t)]\\ \dot{v}(\tau) &= \Pi^2\,v(\tau)\;\;[u(\tau) - 1]\end{aligned}$$

b) Show that the EOM for this biological system has fixed points at $(0,0)$ and $(1,1)$. How does the population model behave close to these fixed points?

c) Sketch the evolution of the solutions in the (u,v)-plane, and compare your result with the data reported on the lynx and hare that are shown in Figure 4.19. Can you find the qualitative difference of the data and behavior predicted by the model? **Hint:** Look at the orientation of the flow in phase space. Who would be eating whom?

d) One can infer the form of the trajectories in phase space by observing that

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\dot{v}}{\dot{u}} = \pi^2\,\frac{v\,(u-1)}{u\,(1-v)}\,.$$

Why does this hold?

e) Find the solution of the ODE by separation of variables and show that the result implies the following constant of motion

$$\Phi(u,v) = \ln\left(v\, u^{\alpha}\right) - v - \alpha\, u\,, \qquad \text{with a suitably chosen } \alpha > 0.$$

Verify this result by also determining the time derivative of $\Phi(u(\tau), v(\tau))$. Here $(u(\tau), v(\tau))$ is a solution of the EOM.

Remark: The presence of a conservation law should be considered an artifact of the model whenever there is no model-immanent (i. e. required by the biological problem in this cases) reason for it to exist.

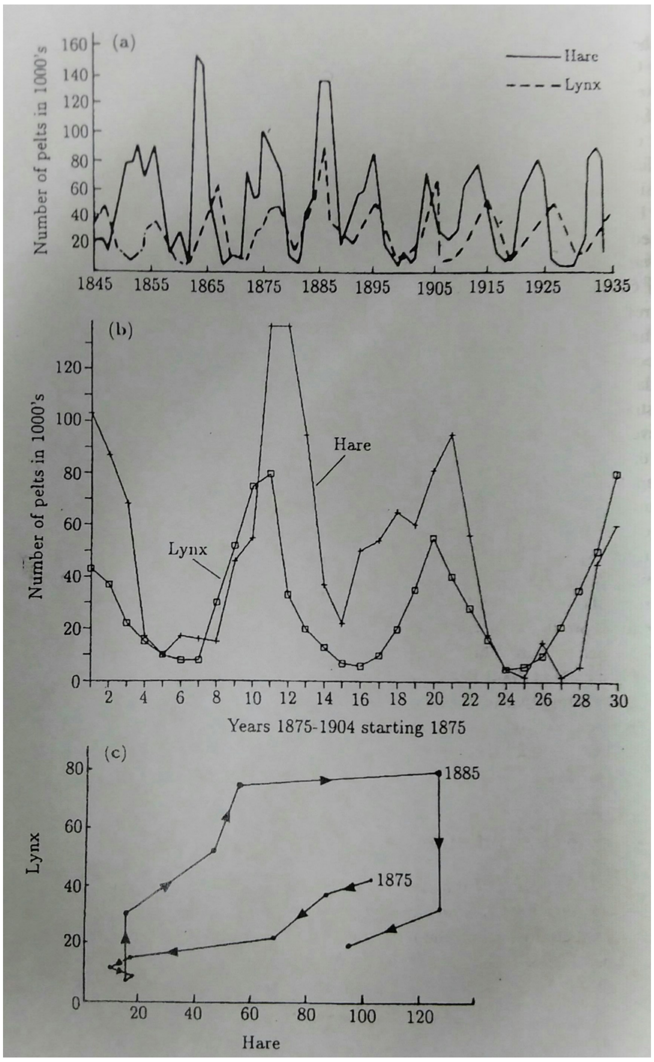


Figure 4.19: (a) Annual oscillations of the skins of hare and lynx offered to the Hudson Bay company. (b) Data with higher time resolution for the 30 years between 1875 and 1904. (c) Presentation of the data presented in (b) as a phase-space plot. [reproduced from Fig. 3.3. of Murray (2002). The book provides a thorough discussion of populations models, their assumptions and artifacts for a range of different populations models.]

grav nut cracker

5
Impact of Spatial Extension

In Chapter 4 we discussed the motion of point particles. However, in our environment the spatial extension of particles in crucial. Physical objects always keep a minimum distance due to their spatial extension. When they had zero extension, one could neither blow up water droplets by impact with a laser (Figure 5.1), nor work clackers (Figure 5.2) or hit a ball with a tennis racket (Figure 5.3). Even giving spin to a ball only works due to the distance between the surface of the racket and the center of the ball.

At the end of this chapter we will be able to discuss the evolution of balls with spin, and their reflections from flat surfaces. Why is spin of so much importance in table tennis? How can a 'Kreisläufer' score a goal in Handball, even when the goal keeper is fully blocking he direct path to the goal? What is the magic of Beckham's banana kicks?

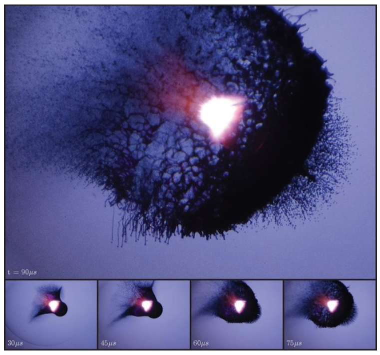


Figure 5.1: Impact of a laser pulse on a microdrop of opaque liquid that is thus blown up; cf. Klein, et al, Phys. Rev. Appl. 3 (2015) 044018

5.1 Motivation and Outline: How do particles collide?

Punt/Anefo, Amsterdam 1971, CC0
Figure 5.2: Girl playing with clackers.

In order to get a first impression about this idea we consider the case of two particles at the positions $\boldsymbol{q}_i$, $i \in \{1, 2\}$ that interact by a repulsive Coulomb force that derives from a potential $\Phi_C(|\boldsymbol{R}|)$ with $\boldsymbol{R} = \boldsymbol{q}_2 - \boldsymbol{q}_1$,

$$\Phi_C(|\boldsymbol{R}|) = \frac{C}{|\boldsymbol{R}|} \quad \Rightarrow \quad \boldsymbol{F}_c(\boldsymbol{q}_i) = -\nabla_{q_i} \Phi_C(|\boldsymbol{q}_i - \boldsymbol{q}_{2-i}|) = \frac{C\, (\boldsymbol{q}_i - \boldsymbol{q}_{2-i})}{|\boldsymbol{q}_i - \boldsymbol{q}_{2-i}|^3}$$

Here, $2 - i$ is the index of the other particle (1 for $i = 2$ and 2 for $i = 1$), and the constant C is the product of the permittivity of the vacuum and the particle charges. For charges of opposite signs this force has agrees with the gravitational force when one substitutes $C \to -m_1\, m_2\, G$. This results in the same dimensionless equations of motion as obtained for the Kepler problem, with the important difference that the length and time units adopted to defined the dimensionless units take vastly different values.

Charlie Cowins from Belmont, NC, USA, CC BY 2.0
Figure 5.3: Man running to return a tennis ball.

When the two particles carry charges of equal signs the force is repulsive, giving rise to the EOM

$$0 = w''(\theta) + w(\theta) + \frac{\mu\, C}{L^2}\, w(\theta)$$

such that

$$R(\theta) = \frac{1}{W(\theta)} = \frac{R_0}{-1 + \epsilon\,\cos\big(\theta - \theta_0\big)} \quad \text{where} \quad R_0 = \frac{L^2}{\mu\, C}$$

agrees with Equation (4.9.3) up to a change of the sign of the one in the denominator and the length unit R_0.

Remark 5.1. It is illuminating to adopt a different perspective on the origin of the minus sign in front of the one. Let us write the force on particle 1 as $\boldsymbol{F}_1 = F_1\,\hat{\boldsymbol{e}}(\theta)$ where $\hat{\boldsymbol{e}}(\theta)$ is the vector pointing from particle 1 to particle 2. The strength of the scalar force F_1 will be positive for an attractive force and negative for a repulsive force. In the dimensionless force $Ft_0^2/\mu\, R_0$ the change of sign is taken into account by the sign of C in $R_0 = L^2/\mu\, C$ and the solution takes the form of Equation (4.9.3). In order to obtain a positive length scale $|R_0| = \pm R_0$ we multiply the numerator and denominator of the solution by the ± 1 and absorb the factor in front of ϵ in a rotation of the angle by π such that the polar coordinates are always aligned with the direction of the force. Hence, one finds

$$R(\theta) = \frac{1}{W(\theta)} = \frac{|R_0|}{\pm 1 + \epsilon\,\cos\big(\theta - \theta_0\big)} \quad \text{where} \quad \pm 1 = \operatorname{sign}(C)$$

At this point dimensionless units play out their strength. We obtain the solution of the nontrivial EOM by an analysis of the ODE and mapping of parameters to a known problem, rather than going again through the involved analysis.

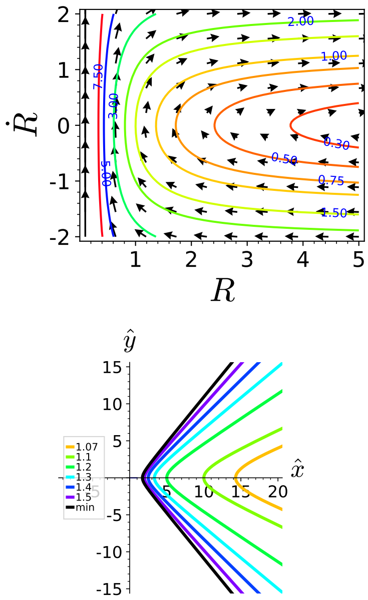


Figure 5.4: Phase-space flow and the shape of trajectories for scattering with a repulsive Coulomb potential.

The phase-space portrait and the shape of the orbits for repulsive interactions are plotted in Figure 5.4. We observe that the trajectory shape describes the approach of the other particle from a perspective of an observer that sits on a particle located in the origin. When the observer sits on a particle that has a much larger mass than the approaching particle, then an outside observer will see virtually no motion of the mass-rich particle and the lines in Figure 5.4 describe the lines of the trajectories of the light particle in a plane selected by the initial angular momentum of the scattering problem. In general, two particles of masses m_1 and m_2 will be at opposite sides of the center of mass. In a coordinate system with its origin at the center of mass the lines in Figure 5.4 describe the particle trajectories up to factors $m_1/(m_1 + m_2)$ and $-m_2/(m_1 + m_2)$ for the first and second particle, respectively. A pair of trajectories for $m_1 = 0.3\,(m_1 + m_2)$ and $\epsilon = 1.2$ is shown in Figure 5.5. The approximation as point particles is well justified when the sum of the particle radii is much smaller than their closest approach $R_0/(\epsilon - 1)$.

Outline

In Section 5.2 we study the collision of spherical hard-ball particles that only interact by a force kick vertical to the surfaces at their contact point when they touch. Then we will compare the Coulomb case and the force-kick case in order to explore which features

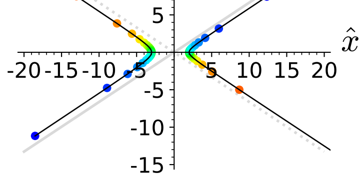


Figure 5.5: The two black lines show the scattering trajectories of two particles with $\epsilon = 1.2$ and relative mass $m_1 = 0.3\,(m_1 + m_2)$. They approach each other along the solid gray line and separate along the dotted line. Particle 1 is initially at the top right. Corresponding positions are marked by dots of matching color.

fill in examples

of the outgoing trajectories are provided by conservation laws, irrespective of the type of interaction. In Section 5.3 we further explore the impact spatial extension and a physical contact of the particles: How does their shape matter and how do balls pick up spin in collisions. Different aspects of this topic will be analyzed in detail by worked examples.

5.1.1 *Self Test*

Problem 5.1. **Scattering angle for the Coulomb potential**

For the choice of coordinates adopted in Figures 5.4 and 5.5 the trajectories have an asymptotic angle θ with the $\hat{x}$-axis when they approach each other and they separate with an asymptotic angle $-\theta$.

a) Show that

$$\tan^2\theta = \frac{2\,E\,L^2}{\mu\,C^2} \tag{5.1.1}$$

b) The parameter dependence of the scattering angle θ is shown in Figure 5.6. What happens to the line for very large values of $2\,E\,L^2/\mu\,C^2$?

c) How would the scattering trajectories in Figure 5.5 look like for $\theta = \pi/2$? Does this comply with your finding in b)?

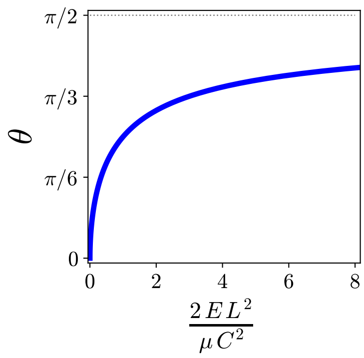


Figure 5.6: Scattering angle θ for a collision of two particles that interact by a repulsive Coulomb potential..

5.2 *Collisions of hard-ball particles*

We consider two spherical particles and denote their radii and masses as R_i and m_i with $i \in \{1,2\}$, respectively. At the initial time $t = t_0$ the particles motion is not (yet) subjected to a force such that

$$\boldsymbol{q}_i(t) = \boldsymbol{q}_i(t_0) + v_i\,(t - t_0)\,, \quad \text{for } \; i \in \{1,2\}$$

5.2.1 *Center of mass motion*

Analogous to the treatment of the Kepler problem, we decompose the motion of the particles into a center-of-mass motion $\boldsymbol{Q}(t)$ and a relative motion $\boldsymbol{r}(t)$. Introducing the notion $M = m_1 + m_2$ the former amounts to

$$M\,\boldsymbol{Q}(t) = m_1\,\boldsymbol{q}_1(t) + m_2\,\boldsymbol{q}_2(t) = M\,\boldsymbol{Q}(t_0) + \dot{\boldsymbol{Q}}(t_0)\,(t - t_0) \tag{5.2.1}$$

Since there are not external forces the total momentum $M\,\dot{\boldsymbol{Q}}(t)$ is conserved (cf. Theorem 3.5) such that Equation (5.2.1) applies for all times – even when the particles collide. A collision will therefore only impact the evolution relative to the center of mass. Equation (5.2.1) holds for all times.

5.2.2 Condition for collisions

To explore the relative motion we write $\boldsymbol{q}_i = \boldsymbol{Q} + \boldsymbol{x}_i$, and we introduce the momentum $\boldsymbol{p} = m_1\,\dot{\boldsymbol{x}}_1 = -m_2\,\dot{\boldsymbol{x}}_2$ and the distance coordinate $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. With these notations the angular momentum of the relative motion reads $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, and it is conserved when the collision force is acting along the line connecting the centers of the particles (cf. Theorem 3.6 and the discussion of Kepler's problem in Section 4.7). Moreover, $\boldsymbol{r}(t)$ is the only time-dependent quantity in this equation because $\boldsymbol{L}$ and $\boldsymbol{p}$ are preserved. Let us first assume that the particles do not collide, and that the closest approach occurs at some time t_c to a distance $r_c = |\boldsymbol{r}(t_c)|$. Then the vectors $\boldsymbol{r}(t_c)$ and $\boldsymbol{p}$ will be orthogonal, and $|\boldsymbol{L}| = r_c\,|\boldsymbol{p}|$. By the properties of the vector product the distance of the closest encounter will always be

$$r_c = \frac{|\boldsymbol{L}|}{|\boldsymbol{p}|} = \frac{\left|m_1\boldsymbol{q}_1(t_0) \times \dot{\boldsymbol{q}}_1(t_0) + m_2\boldsymbol{q}_2(t_0) \times \dot{\boldsymbol{q}}_2(t_0) - M\boldsymbol{Q}(t_0) \times \dot{\boldsymbol{Q}}(t_0)\right|}{m_1\left|\dot{\boldsymbol{q}}_1(t_0) - \dot{\boldsymbol{Q}}\right|}$$

and there will be no collision if $r_c > R_1 + R_2$.

5.2.3 The collision

Conservation of angular momentum implies that the relative motion of the particles proceeds in a plain. When they collide they will approach until, at time t_c, they will reach a position $\boldsymbol{r}(t_c)$ where their distance is $|\boldsymbol{r}(t_c)| = R_1 + R_2$. We denote the direction of $\boldsymbol{r}$ at this time as $\hat{\boldsymbol{\beta}}$ and augment it by an orthogonal direction $\hat{\boldsymbol{\alpha}}$ such that $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{L}} = \boldsymbol{L}/\,|\boldsymbol{L}|)$ form an orthonormal basis. We select the origin of the associated coordinate system such that

$$\boldsymbol{p} = (\boldsymbol{p}\cdot\hat{\boldsymbol{\alpha}})\,\hat{\boldsymbol{\alpha}} + (\boldsymbol{p}\cdot\hat{\boldsymbol{\beta}})\,\hat{\boldsymbol{\beta}}$$

At the collision there is a force $\boldsymbol{F} = F\,\hat{\boldsymbol{\beta}}$ acting on the particles, that acts in the direction of the line $\boldsymbol{r}(t_c)$ connecting the particles. Hence,

1. the momentum component in the $\hat{\boldsymbol{\alpha}}$ direction is preserved during the collision because there is no force acting in this direction
2. the collision in $\hat{\boldsymbol{\beta}}$ direction proceeds like a one-dimensional collision, Example 3.11, with the exception that one must retrace the argument using the center-of-mass frame, as discussed in Problem 4.27.

Consequently, we obtain the following momentum $\boldsymbol{p}'$ after the collision

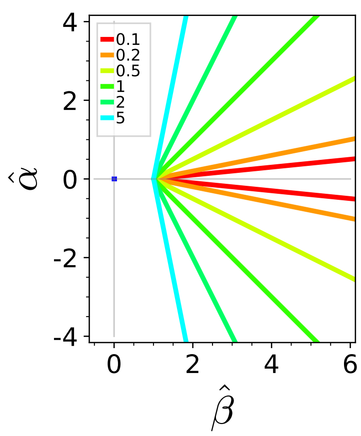
$$\boldsymbol{p}' = (\boldsymbol{p}\cdot\hat{\boldsymbol{\alpha}})\,\hat{\boldsymbol{\alpha}} - (\boldsymbol{p}\cdot\hat{\boldsymbol{\beta}})\,\hat{\boldsymbol{\beta}} = \boldsymbol{p} - 2\,(\boldsymbol{p}\cdot\hat{\boldsymbol{\beta}})\,\hat{\boldsymbol{\beta}}$$

5.2.4 Self Test

Problem 5.2. **Scattering angle for hard-ball particles**

In Figure 5.7 we show shows the trajectory shape and the scattering angle for hard-ball scattering.

a) What is the dimensionless length scale adopted to plot the trajectory shapes?



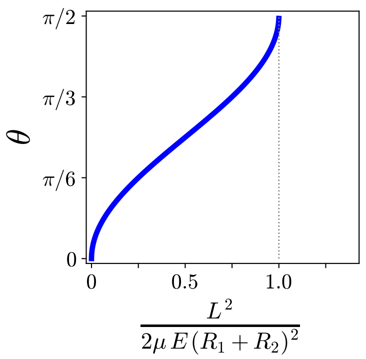


Figure 5.7: Collision of two hard ball particles with radii R_1 and R_2: (top) Trajectory shape. The labels denote

b) What is the impact of the angular momentum on the trajectory shape?
What is the impact of the energy?

c) Verify that

$$\sin^2\theta = \frac{L^2}{2\mu\, E\, (R_1 + R_2)^2} \tag{5.2.2}$$

and that this dependence is plotted in the lower panel of Figure 5.7.

d) What happens when $L^2 > 2\mu\, E\, (R_1 + R_2)^2$?
Which angle θ will one observe in that case?

e) Show that Equations (5.1.1) and (5.2.2) agree when one identifies the length scale $R_1 + R_2$ of the hard-ball system with the distance R_{eff} of symmetry point of the cone section from the origin, i. e. with the mean value of the two intersection points with the $\hat{x}$-axis

$$R_{\text{eff}} = \frac{1}{2}\left(\frac{R_0}{1+\epsilon} + \frac{R_0}{1-\epsilon}\right) = \frac{\epsilon\, R_0}{1-\epsilon^2}$$

Can you provide a physical argument why that must be true?

Problem 5.3. **Reflection from a wall**

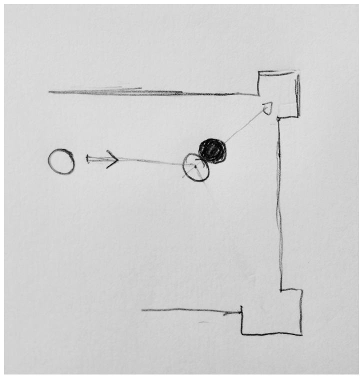
Show that a particle reflected at a flat wall follows the same trajectory as a particle that collides with a particle of the same mass and at a position obtained as mirror image of the particle.

Problem 5.4. **Collisions on a billiard table.**

The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity $\boldsymbol{v}$), and hit the black ball such that it ends up in pocket to the top right.

What is tricky about the sketched track?

What might be a better alternative?



5.3 *Accounting for particle extension*

The center of mass of a set of particles was defined in Equation (4.6.1) as a weighted sum of their positions. Let us now consider an extended object that is characterized by a mass distribution $\rho(\boldsymbol{q})$ with $\boldsymbol{q} = (x, y, z) \in \mathbb{R}^3$. The mass $m(V)$ contained in a volume $\mathcal{V}$ can then be obtained as an integral

$$\begin{aligned} m(V) &= \int_{\mathcal{V}} \mathrm{d}^3 q\, \rho(\boldsymbol{q}) = \int_{\mathcal{V}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \rho(q_1, q_2, q_3) \\ &= \int_{x_{\min}}^{x_{\max}} \mathrm{d}x \left[\int_{y_{\min}(x)}^{y_{\max}(x)} \mathrm{d}y \left(\int_{z_{\min}(x,y)}^{z_{\max}(x,y)} \mathrm{d}z\, \rho(x,y,z)\right)\right] \end{aligned}$$

where the integration runs over all $\boldsymbol{q} \in \mathcal{V}$, a volume with smallest x-value $x_{\min}$ and largest x-value $x_{\max}$, its y-values between $y_{\min}(x)$ and $y_{\max}(x)$ for a given x, and z-values between $z_{\min}(x, y)$ and $y_{\max}(x, y)$ for given x and y.[1] The mass density is zero outside the object such that its total M mass is obtained as

[1] A choice of coordinates that provides simple expressions for the integration boundaries can severely reduce the effort needed to perform the integrals.

$$M = \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})$$

Moreover, the center of mass $\boldsymbol{Q} = (Q_x, Q_y, Q_z)$ of the object is given by the immediate generalization of Equation (4.6.1) ,

$$\boldsymbol{Q} = \frac{1}{M} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \boldsymbol{q} \quad \Leftrightarrow \quad \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \begin{pmatrix} M^{-1} \int_{\mathbb{R}^3 V} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_x \\ M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_y \\ M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_z \end{pmatrix}$$

5.3.1 *The force of an extended object on a point particle*

As a first step towards discussing extended objects we consider the force exerted by an extended object on a point particle. The force is obtained by integrating the forces originating from the mass elements of the body,

$$\boldsymbol{F}_{\text{tot}} = \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \boldsymbol{F}(\boldsymbol{q})$$

where $\boldsymbol{q}$ is the vector from the position of the point particle to the mass element that is exerting the force.

The consequences can nicely be explored when an evil witch switches off electromagnetic interactions between a physics professor and its environment. In the absence of interaction with other matter the professor will freely fall towards the center of Earth, accelerated by a force that arises as sum of the mass elements constituting Earth. For the professor of mass m at position $\boldsymbol{q}_P$ and the mass element at position $\boldsymbol{q}_e$ this force amounts to $\boldsymbol{F}(\boldsymbol{q}_P, \boldsymbol{q}_e) = -\nabla(m\, \rho(\boldsymbol{q}_e)\, G) / |\boldsymbol{q}_P - \boldsymbol{q}_e|$. For simplicity we assume that Earth is spherical and that its mass density takes a uniform value $\rho = 3\, M_E / 4\pi\, R^3$. Then, the force on the professor takes the form

$$\boldsymbol{F}_{\text{tot}} = -\int_{\mathbb{R}^3} \mathrm{d}^3 q\, \nabla \frac{m\, \rho(\boldsymbol{q}_e)\, G}{|\boldsymbol{q}_P - \boldsymbol{q}_e|} \tag{5.3.1}$$

$$= -m\, \rho\, G\, \nabla \int_{\text{Earth}} \mathrm{d}^3 q\, \frac{1}{\sqrt{q_P^2 + q_e^2 - 2\, q_P\, q_e\, \cos\theta}} \tag{5.3.2}$$

where θ is the angle between the two vectors $|\boldsymbol{q}_P|$ and $|\boldsymbol{q}_e|$, while q_P and q_e denote their respective length.

more explanation on integration volumes

The integral is best evaluated by adopting a spherical coordinates (r, θ, ϕ) for the integration where r runs from zero to the Earth radius R, the angle θ from zero to π, and ϕ all around from zero to 2π,

$$
\begin{aligned}
\boldsymbol{F}_{\text{tot}} &= -m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, r^2 \int_0^\pi \mathrm{d}\theta\, \sin\theta \int_0^{2\pi} \mathrm{d}\phi\, \frac{1}{\sqrt{q_P^2 + r^2 - 2\,q_P\,r\,\cos\theta}} \\
&= -2\pi\,m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, r^2 \left[\frac{-1}{q_P\,r}\,\sqrt{q_P^2 + r^2 - 2\,q_P\,r\,\cos\theta}\right]_{\cos\theta=-1}^{\cos\theta=1} \\
&= -2\pi\,m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, \frac{r}{q_P}\, \Big(|q_P + r| - |q_P - r|\Big) \\
&= -4\pi\,m\,\rho\,G\,\nabla \left[\frac{1}{q_P}\,\int_0^{q_P} \mathrm{d}r\, r^2 + \int_{q_P}^R \mathrm{d}r\, r\right] \\
&= m\,\rho\,G\,\nabla \left[2\pi R^2 - \frac{2\pi}{3}\boldsymbol{q}_P \cdot \boldsymbol{q}_P\right] = -\frac{m\,g}{R}\,\boldsymbol{q}_P
\end{aligned}
$$

In the last step we used that the acceleration on the Earth surface is $g = MG/R = 4\pi\,R^2\,G/3$. The professor moves under the influence of a *harmonic* central force, as studied in Problems 4.17, 4.19 and 4.28!

5.3.2 *The force of a point particle on an extended object*

Let us now check which force the point particle exerts on an extended body. The force is described by an integral that takes exactly the same form as Equation (5.3.1), where now $\boldsymbol{q}$ is a vector from the point particle to a volume element of the extended particle.

The integral can be evaluated by introducing a coordinate frame $\hat{\boldsymbol{e}}_1(t), \dots, \hat{\boldsymbol{e}}_3(t)$ with orientation fixed in the rotating body and origin in its center of mass. A given mass element will always have the same coordinates (r_1, r_2, r_3) with respect to this basis, and in a stationary coordinate frame this position can be specified as

$$
\boldsymbol{q}(t) = \boldsymbol{Q}(t) + \sum_{i=1}^{3} r_i\,\hat{\boldsymbol{e}}_i(t) \quad \Rightarrow \quad \ddot{\boldsymbol{q}}(t) = \ddot{\boldsymbol{Q}}(t) + \sum_{i=1}^{3} r_i\,\ddot{\hat{\boldsymbol{e}}}_i(t)
$$

and the force on a spatially extended body results in

$$
\begin{aligned}
\boldsymbol{F}_{\text{tot}} &= \int_{\mathbb{R}^3} \mathrm{d}^3q\, \rho(\boldsymbol{q})\, \ddot{\boldsymbol{q}}(t) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3q\, \rho(\boldsymbol{q})\, \left(\ddot{\boldsymbol{Q}}(t) + \sum_{i=1}^{3} r_i\,\ddot{\hat{\boldsymbol{e}}}_i(t)\right) \\
&= M\,\ddot{\boldsymbol{Q}} + \sum_{i=1}^{3} \ddot{\hat{\boldsymbol{e}}}_i(t)\, \int_{\mathbb{R}^3} \mathrm{d}^3r\, \rho(\boldsymbol{r})\, r_i = M\,\ddot{\boldsymbol{Q}}
\end{aligned}
$$

The overall force $\boldsymbol{F}_{\text{tot}}$ results in an acceleration of the center of mass that behaves exactly as for a point particle described in the previous chapter. Thus, we have justified the assumption of point particles adopted in Chapter 4.

5.3.3 *Particle spin*

The new feature of a spatially extended particles is that they can spin.

Definition 5.1: Particle Spin

The angular momentum of the relative motion of a particle around its center of mass is denoted as *particle spin*, or more concisely as *spin*, $\boldsymbol{S}$. For a body-fixed coordinate system $\{\hat{\boldsymbol{e}}_1(t), \ldots, \hat{\boldsymbol{e}}_3(t)\}$ it takes the form

$$\boldsymbol{S} = \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1.r_2.r_3)\, \boldsymbol{r} \times \dot{\boldsymbol{r}} \quad \text{with} \quad \boldsymbol{r} = \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t) \tag{5.3.3}$$

Remark 5.2. The vector $\boldsymbol{r}$ describes the position $(r_1, \ldots, r_3)$ in the body with respect to its center of mass. When the body rotates $\boldsymbol{r}$ will evolve in time. However, the coordinates $(r_1, \ldots, r_3)$ are constant numbers describing the shape of the body when they are calculated in a coordinate system with base vectors $\{\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_3\}$ fixed in the body and origin in its center of mass. Hence,

$$\dot{\boldsymbol{r}} = \sum_{i=1}^{3} r_i\, \dot{\hat{\boldsymbol{e}}}_i(t)$$

The spin changes in time according to the differential equation

$$\begin{aligned} \dot{\boldsymbol{S}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \times \ddot{\boldsymbol{r}} \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \times \ddot{\boldsymbol{q}} = \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{F}(\boldsymbol{r}) \end{aligned}$$

In order to arrive at the second line, we noted that $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{q}} - \ddot{\boldsymbol{Q}}$, and that the integral for the $\ddot{Q}$ contribution vanishes because $\int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} = 0$. Moreover, it is understood that the force $\boldsymbol{F}$ is zero for coordinates $\boldsymbol{r}$ outside the body.

Definition 5.2: Particle Torque

When the part $\boldsymbol{r}$ of a body is subjected to force $\boldsymbol{F}$ then its spin $\boldsymbol{S}$ is changing due to a torque $\boldsymbol{M}$

$$\dot{\boldsymbol{S}} = \boldsymbol{M} = \int_{\text{body}} \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{F}(\boldsymbol{r})$$

Remark 5.3. Note that the torque is denoted by the letter capital M that is also frequently used for the mass. Nevertheless, there is no immediate risk that they are mixed up: The torque, $\boldsymbol{M}$, is a vector, while the mass, M, is a scalar. To further reduce the risk we will denote masses by a small letter m, when mass and torque appear in a problem.

Remark 5.4. The gravitational force $\boldsymbol{F}(r_1, r_2, r_3) = \rho(r_1, r_2, r_3)\, \boldsymbol{g}$ does not impact the spin:

$$\int_{\text{body}} \mathrm{d}^3 r\, \boldsymbol{r} \times \rho(r_1, r_2, r_3)\, \boldsymbol{g} = \left(\int_{\text{body}} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \right) \times \boldsymbol{g} = \boldsymbol{0} \times \boldsymbol{g} = \boldsymbol{0}$$

The importance of spin is that angular momentum conservation implies that the sum of the spin and the angular momentum of the center-of-mass motion is conserved. As a consequence, the incoming and outgoing angle can differ for a reflection at a wall, and the center of mass of the particle can even move in different planes before and after the collision. This is demonstrated in the following exercises.

5.3.4 Self Test

Problem 5.5. **Reflecting balls**

We consider the reflection of a ball from the ground, the lower side of a table, and back. The ball is considered to be a sphere with radius R, mass m, and moments of inertia $m\alpha R^2$ (by symmetry they all agree). Its velocity at time t_0 will be denoted as $\dot{\boldsymbol{z}}_0$. It has no spin initially. $\boldsymbol{\omega}_0 = \mathbf{0}$. The velocity and the spin after the n^{th} collision will be denoted as $\dot{\boldsymbol{z}}_n$ and $\boldsymbol{\omega}_n$. We will disregard gravity such that the ball travels on a straight path in between collisions.

a) Sketch the setup, and the parameters adopted for the first collision: The positive x axis will be parallel to the floor and the origin will be put into the location of the collision. Its direction will be chosen such that the ball moves in the x-z plane. Take note of all quantities needed to discuss the angular momentum with respect to the origin.

b) Upon collision there is a force normal to the floor, $\boldsymbol{F}_\perp$, and a force tangential to the floor, $\boldsymbol{F}_\parallel$. The spin of the ball will *only* change due to the tangential force. The normal force $\boldsymbol{F}_\perp$ acts in the same way as for point particles. The velocity in vertical direction reverses direction and preserved its modulus. Denote the velocity component in horizontal direction as $v_n = \hat{x} \cdot \dot{\boldsymbol{z}}$, and demonstrate that conservation of energy and angular momentum imply that

$$v_n^2 + \alpha R^2 \omega_n^2 = v_{n+1}^2 + \alpha R^2 \omega_{n+1}^2$$

$$v_n - \alpha R\, \omega_n = v_{n+1} - \alpha R\, \omega_{n+1}\,.$$

Show that the tangential velocity component will therefore also reverse its direction and preserves the modulus,

$$v_n + R\, \omega_n = -(v_{n+1} + R\, \omega_{n+1})\,.$$

c) Determine $v_1(v_0, \omega_0)$ and $\omega_1(v_0, \omega_0)$ for the initial conditions specified above. Now, we determine $v_2(v_1, \omega_1)$ and $\omega_2(v_1, \omega_1)$ by shifting the origin of the coordinate systems to the point where the next collision will arise, and we rotate by π to account for the fact that we collide at the lower side of the table. What does this imply for v_1 and ω_1? Continue the iteration, and plot v_1, v_2 and v_3 as function of α. Discuss the result for a sphere with uniform mass distribution (what does this imply for ω?), and a sphere with $\omega = 1/3$.
Hint: For the plot one conveniently implements the recursion, rather than explicitly calculating v_3.

d) What changes in this discussion when the ball has a spin initially?

5.4 Spinning during flight

5.4.1 The tennis racket theorem — a worked example

add worked example

5.5 Problems

5.5.1 Practicing Concepts

Problem 5.6. **Determining the volume, the mass, and the center of mass**

Determine the mass M, the area or volume V, and the the center of mass $\boldsymbol{Q}$ of bodies with the following mass density and shape.

a) A triangle in two dimensions with constant mass density $\rho = 1\,\mathrm{kg/m}^2$ and side length $6\,\mathrm{cm}$, $8\,\mathrm{cm}$, and $10\,\mathrm{cm}$.
Hint: Determine first the angles at the corners of the triangle. Decide then about a convenient choice of the coordinate system (position of the origin and direction of the coordinate axes).

b) A circle in two dimensions with center at position (a, b), radius $R = 5\,\mathrm{cm}$, and constant mass density $\rho = 1\,\mathrm{kg/m}^2$.
Hint: How do M, V and $\boldsymbol{Q}$ depend on the choice of the origin of the coordinate system?

c) A rectangle in two dimensions, parameterized by coordinates $0 \leq x \leq W$ and $0 \leq y \leq B$, and a mass density $\rho(x, y) = \alpha\, x$.
What is the dimension of α in this case?

d) A three-dimensional wedge with constant mass density $\rho = 1\,\mathrm{kg/m}^3$ that is parameterized by $0 \leq x \leq W$, $0 \leq y \leq B$, and $0 \leq z \leq H - Hx/W$.
Discuss the relation to the result of part b).

e) A cube with edge length L. When its axes are aligned parallel to the axes $\hat{x}, \hat{y}, \hat{z}$, it density takes the form $\rho(x, y, z) = \beta\, z$.
What is the dimension of β in this case?

5.5.2 Proofs

5.5.3 Transfer and Bonus Problems, Riddles

Problem 5.7. **Maximum distance of flight.**

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

c) What is the optimum choice of θ for the shot put?

d) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

6
Lagrange Formalism

In Chapter 5 we considered objects that consist of a mass points with fixed relative positions, like a flying and spinning ping-pong ball. Rather than providing a description of each individual mass element, we established equations of motion for their center of mass and the orientation of the body in space. From the perspective of the theoretical mechanics the fixing of relative positions is a constraint to their motion, just as the ropes of a swing enforces a motion on a one-dimensional circular track, rather than in two dimensions. The deflection angle θ of the pendulum, and the center of mass and orientation of the ball are examples of generalized coordinates that automatically take into account the constraints.

In this chapter we discuss how to set up generalized coordinates and how to find the associated equations of motion. The discussion will be driven by examples. At the end of the chapter you know why coins run away rolling on their edge, and how the speed of a steam engine was controlled by a mechanical device.

Marguerite Martyn, 1914
wikimedia/public domain
Figure 6.1: The point-particle idealization of a girl on a swing is the mathematical pendulum of Figures 1.2 and 1.3.

6.1 Motivation and Outline: How to deal with constraint motion?

Almost all interesting problems in mechanics involve constraints due to rails or tracks, and due to mechanical joints of particles. The most elementary example is a swing (Figure 6.1), where a rope forces a mass M to move on a path with positions constrained to a circle with radius given by the length L of the rope. Gravity $M\,g$ and the pulling force $\boldsymbol{F}_r$ of the rope acting act on the mass (Figure 6.2). However, how large is the latter force? At the topmost point of its trajectory the mass is at rest, and no force is needed along the rope to keep it on its track. At the lowermost point, where the swing goes with its maximum speed, there is a substantial force. Newton's formalism requires a discussion of these forces. Lagrange established an alternative approach that provides equations of motion with substantially less effort. The key idea of this formalism is to select generalized coordinates adapted to the problem.

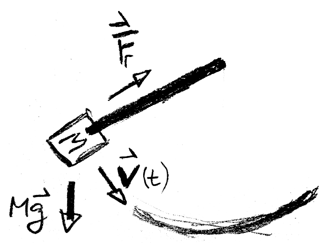


Figure 6.2: Forces acting for the motion on a swing (top), or its equivalent idealization of a mathematical pendulum (bottom).

Definition 6.1: Generalized Coordinates

We consider N particles moving in D dimensions. Their motion is constrained to lie on a prescribed track and their relative positions may be constrained by bars and joints. Due to the constrains the system only has $M < D\,N$ degrees of freedom. In this chapter we denote the positions of the particles as $\boldsymbol{x} \in \mathbb{R}^{D\,N}$, and we specify position compatible with the constraints as $\boldsymbol{x}(\boldsymbol{q}(t))$, where $\boldsymbol{q} \in \mathbb{R}^{M}$ are the *generalized coordinates* adapted to the constrained motion.

Example 6.1: Generalized coordinates for a pendulum

We describe the position of the mass in a mathematical pendulum by the angle $\theta(t)$, as introduced in Example 1.10. The position of the mass in the 2D pendulum plane is thus described by the vector

$$\boldsymbol{x}(t) = L \begin{pmatrix} \sin\theta(t) \\ -\cos\theta(t) \end{pmatrix} = L\,\hat{\boldsymbol{R}}(\theta(t))$$

and its velocity amounts to

$$\dot{\boldsymbol{x}} = L\,\dot\theta\,\partial_\theta \hat{\boldsymbol{R}}(\theta(t)) = L\,\dot\theta\,\hat{\boldsymbol{\theta}}(\theta(t)) \quad \text{with} \quad \hat{\boldsymbol{\theta}}(\theta(t)) = \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}$$

Note that $\hat{\boldsymbol{R}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$ are orthonormal vectors that describe the position of the mass in terms of polar coordinates rather than fixed-in-space Cartesian coordinates.

Example 6.2: Generalized coordinates for a ping-pong ball

A ping-pong ball consists of N atoms located in the three-dimensional space. During a match it they follow an intricate trail in the vicinity of the ping-pong players. At any time during their motion the atoms are located on a thin spherical shell with fixed positions with respect to each other. Rather than specifying the positions of each atoms one can therefore specify the position of the ball in terms of six generalized coordinates:[1] Three coordinates provide its center of mass. The orientation of the ball can be provided by specifying the orientation of a body fixed axis in terms of its polar and azimuthal angle, and a third angle specifies the orientation of a point on its equator when rotating the ball around the axis.

[1] Six coordinates are required when the ball is modeled as a rigid object. Otherwise, additional coordinates are required to describe its deformation.

Generalized coordinates describe only positions complying with the constraints of the motion, and they do not account for other positions by the very beginning. Lagrange's key observation is that constraint forces, e. g. the force on the rope of the swing, only act in a direction orthogonal to the positions described by generalized coordinates. Therefore, the constraint forces do not affect the time evolution of generalized coordinates. For the pendulum and the ping-pong ball one only has to account for gravity to find the evolution of the generalized coordinates. There is no need to deal with the force along the rope in the swing, and the atomic interaction forces that keep atoms in their positions in the ping-pong ball.

Outline

In Section 6.2 we will first present the formalism in action in order to learn *how it works*. Subsequently, in Section 6.3 we explore *why it works*. In the final Section 6.4 we learn how it has impacted the physical world view in the course of the 20$^{\text{Thu}}$ century.

6.2 *Applying the Lagrange formalism*

Let $\boldsymbol{q}$ be the generalized coordinates of a system and $\boldsymbol{x}(\boldsymbol{q})$ the associated configuration vector of the system. Note that $\boldsymbol{x}$ is a vector with all properties discussed in Chapter 2, while $\boldsymbol{q}$ might only be a tuple of functions that provide a convenient parameterization of valid configurations. We address the situation where the forces in the system are conservative, arising from a potential energy $U\big(\boldsymbol{x}(\boldsymbol{q})\big)$. The equations of motion are derived from the Lagrange function, Definition 6.2, by Algorithm 6.1.

Definition 6.2: Lagrange function

The Lagrange function $\mathcal{L}$ amounts to the difference of the kinetic energy T and the potential energy U of the system,

$$\mathcal{L} = T - U = \sum_\alpha \frac{m_\alpha}{2} \dot{x}_\alpha^2(\boldsymbol{q}) - U\big(\boldsymbol{x}(\boldsymbol{q})\big) \tag{6.2.1}$$

Algorithm 6.1: Euler Lagrange EOMs

a) Identify generalized coordinates $\boldsymbol{q}$ that describe the admissible configurations of the system.

b) Determine $\boldsymbol{x}(\boldsymbol{q})$, and the resulting expression of the potential energy in terms of $\boldsymbol{q}$,

$$U(\boldsymbol{q}) = U\big(\boldsymbol{x}(\boldsymbol{q})\big)$$

c) Evaluate the kinetic energy

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_\alpha \frac{m_\alpha}{2} \dot{x}_\alpha^2(\boldsymbol{q}) = \sum_\alpha \frac{m_\alpha}{2} \left(\sum_i \frac{\partial x_\alpha}{\partial q_i}\, \dot{q}_i \right)^2$$

where x_α is the α-component of the configuration vector $\boldsymbol{x}$ and m_α the mass of the associated particle.

Hence, we establish the Lagrange function $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q})$ expressed in terms of the generalized coordinates $\boldsymbol{q}$ and the time derivatives $\dot{\boldsymbol{q}}$.

d) Determine the EOM for the component q_i of $\boldsymbol{q}$ by evaluating the *Euler-Lagrange equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{6.2.2}$$

We will now illustrate the application of the Lagrange formalism for three examples: the mathematical pendulum, Example 6.1, the spherical pendulum, and the motion of a pearl on a rotating ring.

6.2.1 *The EOM for the mathematical pendulum*

The parameterization introduced in Example 6.1 provides the kinetic energy

$$T = \frac{M}{2}\,\dot{\boldsymbol{x}}^2 = \frac{M}{2}\,L^2\,\dot{\theta}^2\,\hat{\boldsymbol{\theta}}(\theta(t))^2 = \frac{M}{2}\,L^2\,\dot{\theta}^2$$

and the potential energy in the gravitational field

$$U = -M\boldsymbol{g}\cdot\boldsymbol{x} = -M\,L\,\hat{\boldsymbol{R}}\big(\theta(t)\big)\cdot\boldsymbol{g} = -M\,L\,g\cos\theta(t)$$

since $\boldsymbol{g} = g\,\hat{\boldsymbol{R}}(0)$.

Consequently,

$$\mathcal{L} = \frac{M}{2}\,L^2\,\dot{\theta}^2 + M\,g\,L\,\cos\theta(t)$$

$$\Rightarrow \quad M\,L^2\,\ddot{\theta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} = -M\,g\,L\,\sin\theta(t)$$

$$\Rightarrow \quad \ddot{\theta}(t) = -\frac{g}{L}\,\sin\theta(t) \tag{6.2.3}$$

The EOM (6.2.3) can be integrated once by multiplication with $2\dot{\theta}(t)$

$$\dot{\theta}^2(t) - \dot{\theta}^2(t_0) = \int_{t_0}^{t} \mathrm{d}t\,2\dot{\theta}\ddot{\theta} = \int_{t_0}^{t}\mathrm{d}t\,2\dot{\theta}\left(-\frac{g}{L}\,\sin\theta(t)\right)$$
$$= 2\int_{\theta(t_0)}^{\theta(t)}\mathrm{d}\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{g}{L}\,\cos\theta\right) = 2\,\frac{g}{L}\,\big(\cos\theta(t) - \cos\theta(t_0)\big)$$

This is a Mattheiu differential equation. For most initial conditions it can not be solved by simple means. However, the first integral provides the phase-space trajectories $\dot{\theta}(\theta)$ for every given set of initial conditions $\big(\theta(t_0), \dot{\theta}(t_0)\big)$,

$$\dot{\theta} = \pm\sqrt{\dot{\theta}^2(t_0) + \frac{2\,g}{L}\,\big(\cos\theta(t) - \cos\theta(t_0)\big)}$$

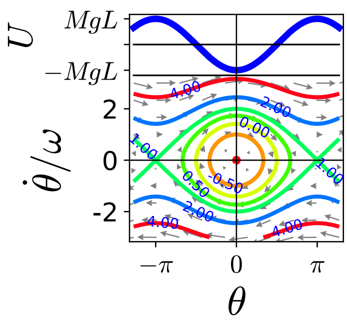


Figure 6.3: The potential $U(\theta)$ (top) and the phase-space plot (bottom) for the EOM (6.2.3) of the mathematical pendulum. The numbers marked on the contour lines indicated the energy of a trajectory in units of MgL.

The phase-space portrait is shown in Figure 6.3. There are trivial solutions where the pendulum is resting without motion at its stable and unstable rest positions $\theta = 0$ and $\theta = \pi$, These positions are denoted as *fixed points* of the dynamics. There are closed circular trajectories close to the minimum, $\theta = 0$, of the potential where it is harmonic to a good approximation. These are solutions with energies $0 < 1 + E/MgL \lesssim 1$.

For larger amplitudes the amplitude of the swinging grows, and the circular trajectories get deformed. When E approaches MgL the phase-space trajectories arrive close to the tipping points $\theta = \pm\pi$ where they form very sharp edges. For θ close to $\theta = \pm\pi$ the trajectories look like the hyperbolic scattering trajectories for the potential $-a\, x^2/2$ that was discussed in Problem 4.23. When the non-dimensional energy is exactly one, the pendulum starts on top, goes through the minimum and returns to the top again. Apart from the fixed points, this is the only case where the evolution can be obtained in terms of elementary functions. For the initial condition $\dot\theta(t_i) = 0$ and $\cos\theta_i = -1$ we find

$$\omega^{-1}\, \dot\theta_H(t) = \pm\sqrt{2 + 2\,\cos\theta_H(t)} = \pm 2\ \cos\frac{\theta_H(t)}{2}$$

The same equation is also obtained for the initial condition $\theta_0 = 0$ and $\dot\theta(t_0) = \sqrt{2g/L}$ half-way on the way from the top back to the top. For this initial condition the ODE for $\dot\theta_H$ can be integrated, and we find

$$\pm 2\,\omega\,(t - t_0) = \int_0^{\theta(t)} \frac{\mathrm{d}\theta}{\cos\frac{\theta}{2}} = \ln\frac{1 + \sin\frac{\theta_H(t)}{2}}{1 - \sin\frac{\theta_H(t)}{2}} - \ln\frac{1}{1}$$

$$\Rightarrow \qquad \theta_H(t) = 2\,\arcsin\tanh\big(\pm\omega\, t\big) \tag{6.2.4}$$

The $\pm$ signs account for the possibility that the pendulum can move clockwise and counterclockwise. The counterclockwise trajectory is shown in Figure 6.4. In the limit $t \to -\infty$ it starts in the unstable fixed point $\theta = -\pi$. It falls down till it reaches the minimum $\theta = 0$ at time t_0, and then it rises again, reaching the maximum $\theta = \pi$ for time $t \to \infty$. Such a trajectory is called a homocline.

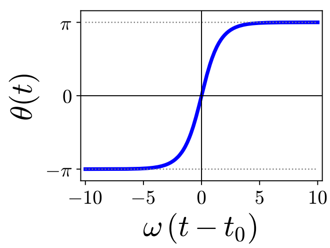


Figure 6.4: Anticlockwise moving heterocline for the mathematical pendulum.

Definition 6.3: Homoclines and Heteroclines

Homoclines and *heteroclines* are trajectories that approach a fixed points of a dynamics in their infinite past and future. A homocline returns to the same fixed point from where it started. A heterocline connects two different fixed points.

The take-home message of this example is that the minima and maxima of a potential organize the phase space flow. Close to each minimum a conservative system will have closed trajectories that represent oscillations in a potential well. The well is confined by maxima to the left and right of the minimum of the potential. When these maxima have different height there is a homoclinic orbit coming down from and returning to the shallower maximum. When they have the same height, they are connected by heteroclinic orbits. Thus, the homoclines and heteroclines divide the phase space into different domains. Initial conditions within the same domain show qualitatively similar dynamics. Initial conditions in different domains feature different dynamics. For the mathematical pendulum the heteroclines divide are three domains, up to the 2π translation symmetry of θ:

a) There are trajectories oscillating around $\theta = 0$, with energies smaller than MgL. The region of these oscillations is bounded by the heteroclines provided in Equation (6.2.4).

b) Trajectories with initial conditions lying above the anticlockwise moving heterocline will persistently rotate anticlockwise and never reverse their motion.

c) Trajectories with initial conditions lying below the clockwise moving heterocline will persistently rotate clockwise and never reverse their motion.

The general strategy for sketching phase-space plots is summarized in the following algorithm.

Algorithm 6.2: Phase space plots

a) Identify the minima and maxima of the potential. Mark the minima as (marginally) stable fixed points with velocity zero. Mark the maxima as unstable fixed points with velocity zero.

b) Identify the fate of trajectories departing from the unstable fixed points. Identify to this end the closest positions on the potential that have the same height as the maximum. When it is another extremum the orbit will form an heterocline. Otherwise, it will be reflected and return to the initial maximum, forming a homocline. If there is no further point of the same height, the trajectory will escape to infinity.

c) Add characteristic trajectories close to the minima and in between homo- and heteroclines.

d) Observe the symmetries of the system. To the very least the plot is symmetric with respect to reflection at the horizontal axis, i. e. swapping the sign of the velocity.

e) Observe energy conservation (if it applies): The modulus of the velocity takes a local minimum for a maximum of the potential, and a local maximum for a minimum of the potential.

6.2.2 *The EOM for the spherical pendulum*

The spherical pendulum describes the motion of a mass M that is mounted on a bar of fixed length ℓ whose other end is fixed to a pivot. Thus, the position of the mass is constraint to a spherical shell. We adopt spherical coordinates to describe the position

$$\boldsymbol{x}(t) = \ell \begin{pmatrix} \sin\theta(t)\ \cos\phi(t) \\ \sin\theta(t)\ \sin\phi(t) \\ -\cos\theta(t) \end{pmatrix} = \ell\, \hat{\mathbf{R}}\big(\theta(t), \phi(t)\big)$$

The angle θ takes values $0 < \theta < \pi$, and it denotes the angle between the position the mass and the gravitational field. Consequently, the potential energy in the gravitational field is obtained

$$U = -M\,\boldsymbol{g}\cdot\boldsymbol{x} = -M\,g\,\ell\,\cos\theta(t)\,.$$

The angle ϕ takes values $0 \le \phi < 2\pi$, and it describes in which direction the mass is deflected from the vertical line, in a plane orthogonal to the action of gravity (see Figure 6.5).

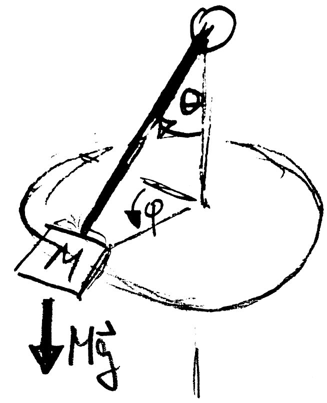


Figure 6.5: Spherical coordinates adopted to describe the motion of a spherical pendulum.

For the velocity we find

$$\begin{aligned}\dot{\boldsymbol{x}} &= \ell\,\dot\theta\,\partial_\theta\hat{\boldsymbol{R}}\big(\theta(t),\phi(t)\big) + \ell\,\dot\phi\,\partial_\phi\hat{\boldsymbol{R}}\big(\theta(t),\phi(t)\big)\\ &= \ell\,\dot\theta\,\hat{\boldsymbol{\theta}}\big(\theta(t),\phi(t)\big) + \ell\,\dot\phi\,\sin\theta(t)\;\hat{\boldsymbol{\phi}}\big(\theta(t),\phi(t)\big)\end{aligned}$$

where we introduced $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ with

$$\hat{\boldsymbol{\theta}}(\theta,\phi) = \begin{pmatrix}\cos\theta\ \cos\phi\\ \cos\theta\ \sin\phi\\ \sin\theta\end{pmatrix} \quad\text{and}\quad \hat{\boldsymbol{\phi}}(\theta,\phi) = \begin{pmatrix}-\sin\theta\ \sin\phi\\ \sin\theta\ \cos\phi\\ 0\end{pmatrix}$$

The unit vectors $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ form a position-dependent orthonormal basis that describes positions in $\mathbb{R}^3$ in terms of polar coordinates. The expression for $\dot{\boldsymbol{x}}$ and $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = 0$ immediately provide the kinetic energy

$$T = \frac{M}{2}\,\dot{\boldsymbol{x}}^2 = \frac{M}{2}\,\ell^2\,\dot\theta^2(t) + \frac{M}{2}\,\ell^2\,\sin^2\theta(t)\,\dot\phi^2(t)$$

Consequently, the Lagrange function for the spherical pendulum takes the form

$$\mathcal{L}(\theta,\phi,\dot\theta,\dot\phi) = \frac{M}{2}\,\ell^2\,\dot\theta^2 + \frac{M}{2}\,\ell^2\,\sin^2\theta(t)\,\dot\phi^2(t) + M\,g\,\ell\,\cos\theta(t)$$

We observe that $\mathcal{L}$ does not depend on ϕ. In that case it is advisable to first discuss the EOM for ϕ. It takes the form

$$M\,\ell^2\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot\phi\,\sin^2\theta(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

The derivative of the Lagrange function with respect to ϕ vanishes because $\mathcal{L}$ does not depend on ϕ. Such a coordinate is called a cyclic, and it always implies a conservation law, C. For the spherical pendulum it signifies conservation of the z-component of the angular momentum, and it provides an expression of $\dot\phi$ in terms of θ

$$C = \dot\phi\,\sin^2\theta(t) = \text{const} \quad\Rightarrow\quad \dot\phi(t) = \frac{C}{\sin^2\theta(t)} \tag{6.2.5}$$

where C is proportional to the z-component of the angular momentum.

The general case is summarized in the following definition:

Definition 6.4: Cyclic coordinates

A coordinate q_i is called *cyclic* when the Lagrange function depends only on its time derivative $\dot{q}_i$, and not on q_i. In that case the associated Euler-Lagrange equation establishes a conservation law,

$$C = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

After all
$$\frac{\mathrm{d}\, C}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Remark 6.1. The constant value of C is determined by the initial conditions on $\dot{q}_i$ and on the other coordinates.

Let us now consider to the other coordinate of the spherical pendulum. The EOM for $\theta(t)$ takes the form

$$\begin{aligned} M\,\ell^2\,\ddot{\theta}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\ &= \frac{\partial \mathcal{L}}{\partial \theta} = M\,\ell^2\,\dot{\phi}^2(t)\;\sin\theta(t)\;\cos\theta(t) - M\,g\,\ell\,\sin\theta(t) \end{aligned}$$

In this equation the unknown function $\dot{\phi}(t)$ can be eliminated by means of the conservation law, Equation (6.2.5), yielding

$$\ddot{\theta}(t) = \frac{C^2\,\cos\theta(t)}{\sin^3\theta(t)} - \frac{g}{\ell}\,\sin\theta(t)$$

and the resulting EOM can be integrated once by multiplication with $2\dot{\theta}(t)$

$$\begin{aligned} \dot{\theta}^2(t) - \dot{\theta}^2(t_0) &= \int_{t_0}^{t} \mathrm{d}t\, 2\dot{\theta} \left(\frac{C^2\,\cos\theta(t)}{\sin^3\theta(t)} - \frac{g}{\ell}\,\sin\theta(t) \right) \\ &= -2 \int_{\theta(t_0)}^{\theta(t)} \mathrm{d}\theta\, \frac{\mathrm{d}}{\mathrm{d}\theta} \left(-\frac{C^2}{\sin^2\theta} + \frac{g}{\ell}\,\cos\theta \right) \end{aligned}$$

The result can be written in the form

$$\begin{aligned} E &= \frac{\dot{\theta}^2}{2} + \Phi_{\text{eff}}(\theta) = \text{const} \\ \text{where} \quad \Phi_{\text{eff}}(\theta) &= \frac{C^2}{\sin^2\theta} - \frac{g}{\ell}\,\cos\theta \end{aligned}$$

Again a closed solution for $\theta(t)$ is out of reach. However, $\Phi_{\text{eff}}(\theta)$ can serve as an effective potential for the 1DOF motion of θ with kinetic energy $\dot{\theta}^2/2$. This interpretation of the dynamics provides ready access to a qualitative discussion of of the solutions of the EOM based on a phase-space plot.

For $C = 0$ the particle swings in a fixed plane selected by $\phi =$ const. Its motion amounts to that of a mathematical pendulum.

Figure 6.6 shows the effective potential and phase space portraits for different positive values of C. Conservation of angular momentum implies that for $C \neq 0$ the particle can no longer access the region close to its rest position at the lowermost point of the

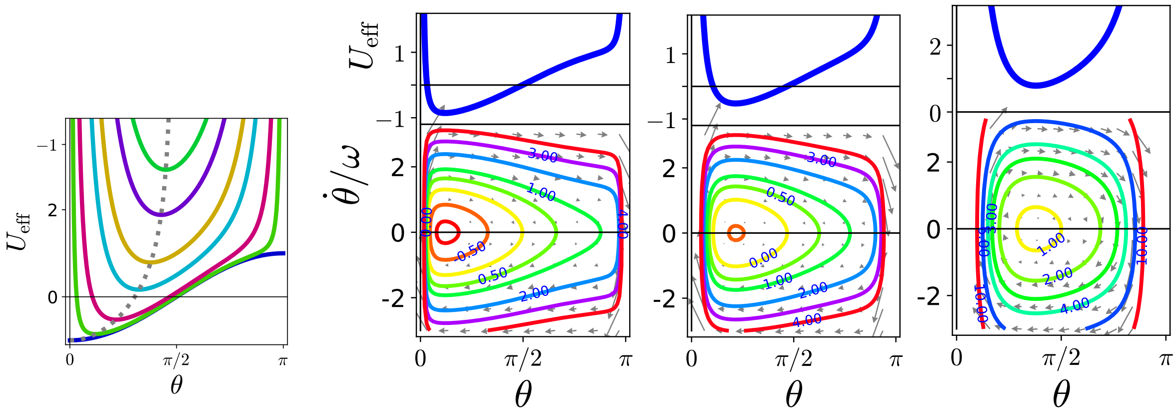


Figure 6.6: The left panel shows the effective potential for the spherical pendulum at parameter values $C^2 \in \{0, 0.01, 0.1, 0.5, 1, 2, 3\}$ from bottom to top. The subsequent panels show phase-space portraits of the motion for $C^2 = 0.01$, 0.1, and 1, respectively.

sphere. Rather it always has to go in circles around the bottom of the well, and the sign of C specifies whether it moves clockwise or anti-clockwise. The divergence of the effective potential at $\theta = \pm\pi$ is called *rotation barrier*. It emerges due to a combination of the conservation of energy and angular momentum.

add problem

The effective potential has a single minimum for $0 < \theta_c(C) < \pi/2$, and not further extrema. The minimum describes motion where the particle moves at constant height with a constant speed in a circle. When this orbit is perturbed oscillations are superimposed on the circular motion. In a projection to the plane vertical to the action of gravity, this will lead to trajectories similar to those drawn by a Spirograph.

add problem

The take-home message of this example is that cyclic variables entail conservation laws of the dynamics. In the very same manner as for the Kepler problem they can be used to eliminate a variable from the EOM of the other coordinates. The additional contributions in the EOMs for the other coordinate(s) are interpreted as part of an effective potential.

6.2.3 The EOM for a pearl on a rotating ring

We consider a pearl of mass M that can freely move on a ring. The ring is mounted vertically in the gravitational field and it spins with angular velocity Ω around its vertical symmetry axis. Again the setup constraints the position of the pearl to lie on a spherical shell, and we hence describe its position as

$$\boldsymbol{x}(t) = \ell\, \hat{\boldsymbol{R}}(\theta(t), \phi(t))$$

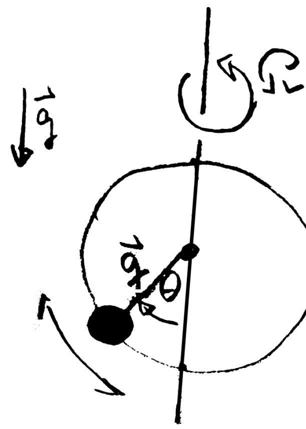


Figure 6.7: Motion of a pearl moving on a ring rotating with a fixed frequency Ω.

However, in this case the position of the pearl is fully described by the angle $\theta(t)$ of the deflection of the pearl from the direction of gravity (see Figure 6.7. The angle $\phi(t) = \Omega t$ is entering the problem as a parameter, dictated by the setup of the problem, and the motion of the pearl on the ring will be described based on a single EOMs for its coordinate $\theta(t)$.

The potential energy takes the same form as for the spherical pendulum. The kinetic energy is obtained based on its velocity

$$\dot{\boldsymbol{x}} = \ell\,\dot\theta\,\hat{\boldsymbol{\theta}}(\theta(t),\Omega t) + \ell\,\Omega\,\sin\theta(t)\;\hat{\boldsymbol{\phi}}(\theta(t),\Omega t)$$

which provides the Lagrange function

$$\mathcal{L}(\theta,\dot\theta) = \frac{M}{2}\,\ell^2\,\dot\theta^2 + \frac{M}{2}\,\ell^2\,\Omega^2\,\sin^2\theta(t) + M\,g\,\ell\,\cos\theta(t)$$

It only differs from the expression for the spherical pendulum by the fact that $\phi(t)$ is not a coordinate whose evolution must be determined from an EOM. Rather it is a parameter $\phi(t) = \Omega\,t$ provided by the setting of the problem.

The motion only has a single DOF, $\theta(t)$, with EOM

$$\ddot\theta(t) = -\frac{g}{\ell}\,\sin\theta(t)\,\left(1 - \frac{\ell\,\Omega^2}{g}\,\cos\theta(t)\right) \tag{6.2.6}$$

This EOM can be integrated by the same strategy adopted for the swing and the spherical pendulum. Thus, one finds the effective potential

$$U_{\text{eff}}(\theta) = -\omega^2\,\cos\theta\,\left[1 - \frac{1}{2}\,\left(\frac{\Omega}{\omega}\right)^2\,\cos\theta\right]$$

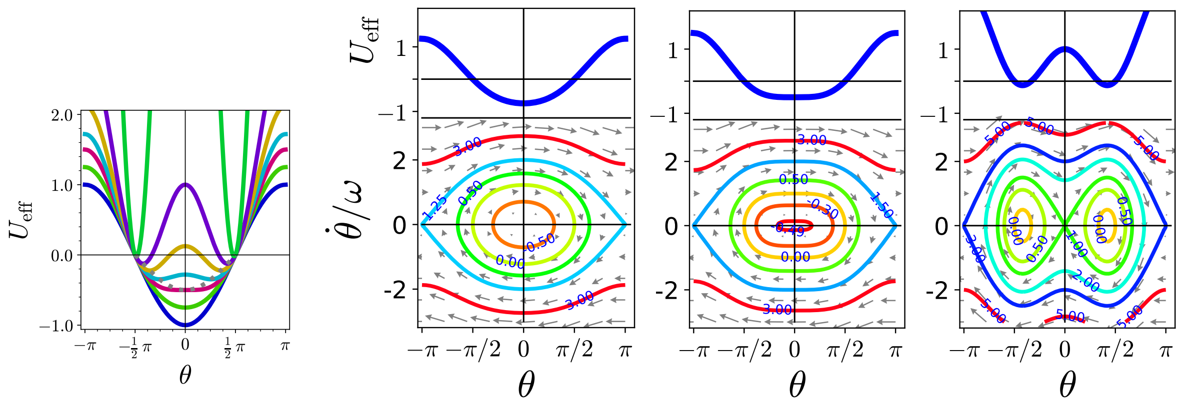


Figure 6.8: The left panel shows the effective potential for the pearl on a ring for parameter values $(\Omega/\omega) \in \{0, 2^{-1/2}, 1, 1.2, 1.5, 2, 5\}$ from bottom to top. The subsequent panels show phase-space portraits of the motion for $\Omega/\omega = 2^{-1/2}$, 1, and 2, respectively.

Figure 6.8 shows the effective potential and phase space portraits for different values of angular momentum, i. e. of the dimensionless control parameter $\kappa = \Omega/\omega$. For $\kappa < 1$ the phase space has the same structure as that of a planar mathematical pendulum, with a stable fixed point at $\theta = 0$. When κ passes through one, this minimum of U_{eff} turns into a maximum, and two new minima emerge at the positions $\theta_c = \pm\arccos\kappa^{-1}$ that are indicated by a dotted gray line in the left panel of Figure 6.8. The new maximum at zero is always shallower than the maxima at $\pm\pi$. Hence, it gives rise to two homoclinic orbit that wind around the new stable fixed points.

The maxima at $\pm\pi$ will further we connected by heteroclinic orbits. Hence, phase space is divided into five distinct regions. For energies smaller than $U_{\text{eff}}(\theta = 0)$ the trajectories wiggle around trajectories that behave similarly to those in the spherical pendulum. They stay on one side of the ring and oscillate around the angle θ_c. There are two regions of this type because the pear can stay on both sides of the ring. For $U_{\text{eff}}(\theta = 0) < E < U_{\text{eff}}(\theta = \pi)$ the trajectories show oscillations back and forth between the two sides of the ring, similar to the mathematical pendulum—except that they slow down in the region $-\theta_s(K) < \theta = 0 < \theta_s(K)$. For $E > U_{\text{eff}}(\theta = \pi)$ they rotate around the ring in clockwise or counter-clockwise direction for $\dot\theta < 0$ or $\dot\theta > 0$, respectively.

There are two take-home message from this example:

1. There are no conservation laws in the dynamics when there are explicitly time-dependent constraints. Hence, the strategies of Chapter 4 to establish and discuss the EOM can no longer be applied, and there also is no rotation barrier. However, the Lagrange formalism still provides the EOM in a straightforward manner.

add problem

2. In general, the structure of the phase-space flow changes upon varying the dimensionless control parameters of the dynamics. These changes are called bifurcations, and they are a very active field of contemporary research in theoretical mechanics. The pearl on the ring features a pitchfork bifurcation We will come back to this topic in due time.

6.2.4 *Self Test*

Problem 6.1. **Phase-space analysis for a pearl on a rotating ring**

a) Verify then by explicit calculation that $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ obey the relations

$$\hat{\boldsymbol{\theta}} = \frac{\partial \hat{\boldsymbol{R}}}{\partial \theta} \qquad \text{and} \qquad \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{R}} \times \hat{\boldsymbol{\theta}}\,,$$

and that they form an orthonormal basis.
How is $\hat{\boldsymbol{\phi}}$ related to $\partial\hat{\boldsymbol{R}}/\partial\phi$?

b) Evaluate $\dot{\boldsymbol{x}}(t) = \ell\,\dot{\hat{\boldsymbol{R}}}(\theta(t), \Omega t)$ based on the relations established in a).

c) Determine the kinetic energy T and the potential energy V of the pearl.

d) Fill in the steps in the derivation of the EOM for θ, as provided in Equation (6.2.6).

e) Determine the fixed points for the motion of the pearl, and discuss their stability as function of Ω.

f) Sketch phase-space plots for the motion of the pearl for $\Omega^2 < g/\ell$ and $\Omega^2 > g/\ell$.

g) How do the phase-space portraits change in the presence of friction?

Problem 6.2. **Kitchen Pendulum**

Figure 6.9: Setup of the kitchen pendulum.

We consider a pendulum that is built from two straws (length L_1 and L_2), two corks (masses m_1 and m_2), a paper clip, and some Scotch tape (see picture to the right). It is suspended from a shashlik skewer, and its motion is stabilized by means of the spring taken from a discharged ball-pen. Hence, the arms move vertically to the skewer. We denote the angle between the arms as α, and the angle of the right arm with respect to the horizontal as $\theta(t)$.

a) Determine the kinetic energy, T, and the potential energy, V, of the pendulum. Argue that T and V can only depend on θ and $\dot{\theta}$, and determine the resulting Lagrangian $\mathcal{L}(\theta, \dot{\theta})$.

b) Determine the EOM of the pendulum.

c) Find the rest positions of the pendulum, and discuss the motion for small deviations from the rest positions. Sketch the according motion in phase space.

d) The EOM becomes considerably more transparent when one selects the center of mass of the corks as reference point. Show that the center of mass lies directly below the fulcrum when the pendulum it at rest.

e) Let ℓ be the distance of the center of mass from the fulcrum, and $\varphi(t)$ be the deflection of their connecting line from the vertical. Determine the Lagrangian $\mathcal{L}(\varphi, \dot{\varphi})$ and the resulting EOM for $\varphi(t)$.

f) Do you see how the equations for $\ddot{\theta}(t)$ and $\ddot{\varphi}(t)$ are related?

6.3 *Understanding the Lagrange formalism*

6.4 *Implications of the Lagrange formalism*

6.5 *Problems*

Take Home Message
Hints for Exam Preparation

The aim of the present course has been to give a first gimps into scientific modeling. It focussed on mechanics problems. Firstly, they are easily visualized. Secondly, they readily provide interesting mathematical challenges when one strives for a comprehensive description. Thus, they provide a unique set of problems to get acquainted with the use of mathematics as a language to address scientific problems. The involved mathematical concepts can further be underpinned in forthcoming mathematics classes. Further physical problems will be addressed in forthcoming experimental and theoretical physics lectures.

What are the next steps to be taken? To begin with you should re-read the script and revisit the exercise sheets in order to prepare for the exam. Take a particular look at exercises that were challenging at the first encounter. In doing so you should focus on understanding the rules of the game, and hands-on application of the mathematical formalism, rather than understanding the concepts in full depth. The concepts might be dealt with again in other classes. Most likely they will not put as much emphasis, however, on practicalities about the careful and efficient setup of the mathematical setup for concrete calculations.

Best wishes, success and fun for your further studies!

A
Physical constants, material constants, and estimates

$$1\,\text{year} \simeq \pi \times 10^7\text{s} \qquad \text{(A.0.1)}$$

A.1 Solar System

The solar system has 1.0014 solar masses, which amounts to 1.991×10^{30} kg.
The Earth-Sun distance is $1\,\text{AU} \simeq 500\,\text{light second} \simeq 1.5 \times 10^{11}$ m.

object	**Sun**	**Mecury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**	**Uranus**	**Neptun**
distance	0.005	0.387098	0.723332	1	1.523679	5.2044	9.5826	19.2184	30.11
radius	109	0.3829	0.9499	1	0.533	11.209	9.449	4.007	3.883
mass	333,000	0.055	0.815	1	0.107	317.8	95.159	14.536	17.147
period		0.240846	0.615198	1	2.1354	11.862	29.4571	84.0205	164.8

Table A.1: Properties of Sun and planets of our solar system, provided in multiples of the Earth values. The distance referes to the semi-major axis in AU. For the sun the distance denotes the sun surface, i. e. its radius.

object	**Moon**	**Ceres**	**Pluto**	**Eris**
distance	0.00257	2.769	39.482	67.864
radius	0.2727	0.073	0.1868	0.1825
mass	0.0123	0.00016	0.00218	0.0028
period	0.08085	4.61	247.94	559.07

Table A.2: Properties of the Moon and dwarf planets of our solar system. The properties of the Moon refer to its distrance to and period around Earth. Ceres is the largest object in the meteorite belt between Mars and Jupiter. Eris is a dwarf planet in the Kuiper belt that is larger in mass than Pluto.

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