Lecture Notes by Jürgen Vollmer

Theoretical Mechanics

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http://www.uni-leipzig.de/physik/~vollmer

Die Philosophie steht in diesem großen Buch geschrieben, dem Universum, das unserem Blick ständig offen liegt. Aber das Buch ist nicht zu verstehen, wenn man nicht zuvor die Sprache erlernt und sich mit den Buchstaben vetraut gemacht hat, in denen es geschrieben ist. Es ist in der Sprache der Mathematik geschrieben, und deren Buchstaben sind Kreise, Dreiecke und andere geometrische Figuren, ohne die es dem Menschen unmöglich ist, ein einziges Wort davon zu verstehen; ohne diese irrt man in einem dunklen Labyrinth herum.

Galileo Galilei, Il Saggiatore, 1623

Die Mathematik ist das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten: sie baut die verbindende Brücke und gestaltet sie immer tragfähiger. Daher kommt es, daß unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Natur beruht, ihre Grundlage in der Mathematik findet.

David Hilbert, Ansprache "'Naturerkennen und Logik"' am 8.9.1930 während des Kongresses der Vereinigung deutscher Naturwissenschafter und Mediziner

Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.

Albert Einstein Festvortrag "'Geometrie und Erfahrung"' am 27.1.1921 vor der Preußischen Akademie der Wissenschaften

Preface

Die ganzen Zahlen hat der liebe Gott geschaffen,
alles andere ist Menschenwerk.
Leopold KRONECKER

Almost 400 years ago GALILEI GALILEO expressed the credo of modern sciences: The language of mathematics is the appropriate instrument to decode the secrets of the universe. Arguably the fruits of this enterprise are more visible today than they have ever been in the past. Mathematical models are the cornerstone of moder science and engineering. They provide the tools for optimizing engines, and the technology for data and communication sciences. No car will run, no plane will fly, no cell phone ring without the technical equipment and the software to make it run. Moreover, ever again the challenges of physics models inspired the development of new mathematics. Indeed, physcis and mathematics take complementary perspectives: Mathematicians strive for a logically stringent representation of the structure of theories and models. Physicists adopt mathematics as a tool speak about and better understand nature.

The present Lecture Notes accompany the course "Theoretical Mechanics" for physics freshmen in the international physics program of the Universität Leipzig. The course addresses mechanics problems to introduce the students to concepts and strategies aiming at a quantitative descrition of observations. Hence, the lectures have several purposes:

a) They introduce the concept of a mathematical model, its predictions, and how they relate to observations.

b) They present strategies adopted to develop a model, to explore its predictions, to falsify models, and to refine them based on comparison to observations.

c) They introduce mathematical concepts used in this enterprise:

dimensional analysis, non-dimensionalization, complex numbers, vector calculus, and ordinary differential equations.

d) They provide an introduction to Newtonial and Lagrangian Mechanics.

Our approach to mathematical concepts is strongly biased to developing the skills to apply the tools in a modeling context, rather than striving for mathematical rigor. For the latter we point out potential pitfalls based on physical examples, and refer the students to forthcoming maths classes. On request and/or need additional topics, that are useful as a reference and to rehearse elementary concepts, can be added in appendices or outlook sections in the chapters.

add reading plan

The material is organized in chapters that address subsequent mathematical and physical topics. Each chapter is introduced by a physcis illustration problem. Then, we develop and discuss relevant new concepts. Subsequently, we provide a worked examples. One of them will be the solution of the problem sketched in the introduction. Finally, there is a section with different types of problems:

a. quickies to test conceptual understanding and highlight the new concepts.

b. exercises to gain practive in employing the concepts.

c. more elaborate exercises where the new concepts are used to discuss non-trivial problems.

add more explanation

d. exercises that provide complementary insight based on Python and Sage programs

e. teasers with challenging problems. Typically these exercises require a non-trivial combination of different concepts that have been introduced in earlier chapters.

acknowledge co-workers

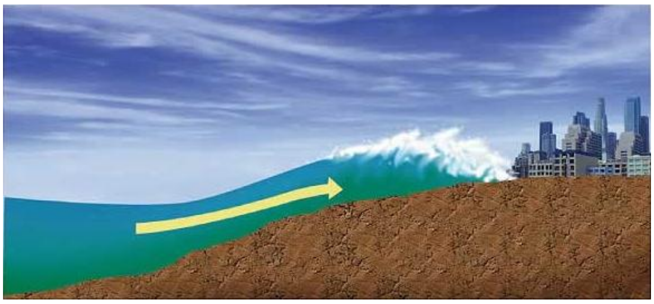
At the end of the chapter we recommend additional literature and provide an outlook for further reading.

I am eager to receive feedback. It will be crucial for the development of this project to learn about typos, inconsistencies, confusing or incomplete explanations, and suggestions for additional material (contents as well as links to papers, books and internet resources) that should be added in forthcoming revisions. Everybody who is willing to provide feedback will be invited to a coffee in Cafe Corso.

Contents

1

Basic Principles



[-Ilhador- Public domain]

At the end of this chapter we will be able to estimate the speed of a Tsunami wave.

1.1 Basic Notions of Mechanics

Definition 1.1: System

A mechanical *system* is comprised of a set of masses m_i, $i \in \mathbb{I}$,
at positions $\vec{x}_i$ that move with velocities $\vec{v}_i$.

Remark 1.1. We say that the system has N particles when $\mathbb{I} = \{1, \ldots, N\}$.

Remark 1.2. The arrows indicate here that $\vec{x}_i$ describes a position in space. For a D space $\vec{x}_i \in \mathbb{R}^D$, and we say that $\vec{x}_i$ is a D-vector.

Remark 1.3. In order to emphasize the close connection between positions and velocities, the latter will also be denoted as $\dot{\vec{x}}$.

Example 1.1: A piece of chalk

We wish to follow the trajectory of a piece of chalk through the lecture hall. In order to follow its position and orientation in space, we *decide* to model it as a set of two masses that are localized at the tip and at the tail of the chalk. The positions of these two masses $\vec{x}_1$ and $\vec{x}_2$ will both be vectors in $\mathbb{R}^3$. For instance we can indicate the shortest distance to three walls that meet in one corner of the lecture hall. In this model we have $N = 2$ and $D = 3$.

Definition 1.2: Degrees of Freedom (DOF)

A system with N particles whose positions are described by D vectors has $D\,N$ *degrees of freedom (DOF)*.

Remark 1.4. Note that according to this definition the number of DOF is a property of the model. For instance, the model for the piece of chalk has $D\,N = 6$ DOF. However, the length of the piece of chalk does not change. Therefore, one can find an alternative description that will only evolve 5 DOF. (We will come back to this in due time.)

Definition 1.3: State Vector

The position of all particles can be written in a single *state vector*, $\vec{q}$, that specifies the positions of all particles.

Remark 1.5. For a system with N particles whose positions are specified by D-dimensional vectors, $\vec{x}_i = (x_{i,1}, \ldots, x_{i,D})$, the vector $\vec{q}$ takes the form $\vec{q} = (x_{1,1}, \ldots, x_{1,D}, x_{2,1}, \ldots, x_{2,D}, \ldots, x_{N,1}, \ldots, x_{N,D})$. For conciseness we will also write $\vec{q} = (\vec{x}_1, \ldots, \vec{x}_N)$. The vector $\vec{q}$ has DOF number of entries.

Remark 1.6. The velocity associated to $\vec{q}$ will be denoted as $\dot{\vec{q}} = (\dot{\vec{x}}_1, \ldots, \dot{\vec{x}}_N)$.

Definition 1.4: Phase Vector

The position and velocities of all particles form the *phase vector*, $\vec{\Gamma} = (\vec{q}, \dot{\vec{q}})$.

Definition 1.5: Trajectory

The *trajectory* of a system is described by specifying the time dependent functions

$$
\begin{aligned}
& \vec{x}_i(t), \vec{v}_i(t), \quad i = 1, \ldots, N \\
\text{or} \quad & \vec{q}(t), \dot{\vec{q}}(t) \\
\text{or} \quad & \vec{Gamma}(t)
\end{aligned}
$$

Definition 1.6: Initial Conditions (IC)

For $t \in [t_0, \infty)$ the trajectory is uniquely determined by its *initial conditions (IC)* for the positions $\vec{x}_i(t_0)$ and velocities $\vec{v}_i(t_0)$,, i.e. the point $\vec{\Gamma}(t_0)$ in phase space.

Remark 1.7. This definition expresses that the future evolution of a system is *uniquely* determined by its ICs. Such a system is called deterministic. Mechanics addresses the evolution of deterministic systems. At some point in your studies you might encounter stochastic dynamics where different rules apply.

Example 1.2: Throwing a javelin

The ICs for the flight of a javelin specify where it is released, $\vec{x}$, when it is thrown, the velocity $\vec{v}$ at that point of time, and the orientation of the javelin. In a good trial the initial orientation is parallel to $\vec{v}$, as shown in Figure 1.1

Figure 1.1: Initial conditions for throwing a javelin, cf. Example 1.2. [Atalanta, creativecommons, CC BY-SA 3.0]

Remark 1.8. In repeated experiments the ICs will be (slightly) different, and one observes different trajectories.

1. A seasoned soccer player will hit the goal in repeated kicks. However, even a professional may miss occasionally.

2. A bicycle involves a lot of mechanical pieces that work together to provide a predictable riding experience.

3. A lottery machine involves a smaller set of pieces where a different (in practice unpredictable) set of balls is selected in each run, in spite of best efforts to select identical initial conditions.

Definition 1.7: Constant of Motion

A function of the positions $\vec{x}_i$ and velocities $\vec{v}_i$ that does not evolve in time is called a *constant of motion*.

Example 1.3: A piece of chalk

During the flight the positions $\vec{x}_1$ and $\vec{x}_2$ of the piece of chalk will change. However, the length L of the piece of chalk will not, and at any given time it can be determined from $\vec{x}_1$ and $\vec{x}_2$. Hence, L is a constant of motion.

Definition 1.8: Parameter

In addition to the ICs the trajectories will depend on *parameters* of the system. Their values are fixed for a given system.

Example 1.4: A piece of chalk

For the piece of chalk the trajectory will depend on whether the hall is the Theory Lecture Hall in Leipzig, a briefing room in a ship during a heavy storm, or the experimental hall of the ISS space station. To the very least one must specify how the gravitational acceleration acts on the piece of chalk, and how the room moves in space.

Remark 1.9. The set of parameters that appear in a model depends on the *choices* that one makes upon setting up the experiment. For instance

Beckham's kicks can only be understood when one accounts for the impact of air friction on the soccer ball.

Air friction will not impact the trajectory of a small piece of talk that I through into the dust bin.

By adopting a clever choice of the parameterization the trajectory of the piece of chalk can be described in a setting with 4 DOF. The length of the piece of chalk will appear as a parameter in that description.

Definition 1.9: Physical Quantities

Positions, velocities and parameters are *physical quantities* that are characterized by at least one numbers and a unit.

Example 1.5: Physical Quantities

1. The mass of a soccer ball can be fully characterized by a number and the unit kilogram (kg), e.g. $M \approx 0.8\,\mathrm{kg}$.
2. The length of a piece of chalk can be fully characterized by a number and the unit meter (m), e.g. $L \approx 7 \times 10^{-2}\,\mathrm{m}$.
3. The length T of a year can be characterized by a number and the unit second, e.g. $T \approx \pi \times 10^{7}\,\mathrm{s}$.
4. The speed of a car can be fully characterized by a number and the unit, e.g. $v \approx 72\,\mathrm{km\,h^{-1}}$.
5. A position in a D-dimensional space can fully be characterized by D numbers and the unit meter.
6. The velocity of a piece of chalk flying through the lecture hall can be characterized by three numbers and the unit m/s.

Remark 1.10. Analyzing the units of the parameters of a system provides a fast way to explore and write down functional dependencies. When doing so, the units of a physical quantity Q are denoted by $[Q]$. For instance for the length L of the piece of chalk, we have $[L] = \mathrm{m}$. For a dimensionless quantity d we write $[d] = 1$.

Example 1.6: Changing units

Suppose we wish to change units from km/h to m/s. A transparent way to do this for the speed of the car in the example above is by multiplications with one

$$v = 72\frac{\mathrm{km}}{\mathrm{h}}\,\frac{1\,\mathrm{h}}{3.6 \times 10^{3}\,\mathrm{s}}\,\frac{1 \times 10^{3}\,\mathrm{m}}{1\,\mathrm{km}} = \frac{72}{3.6}\mathrm{m\,s^{-1}} = 20\,\mathrm{m\,s^{-1}}$$

Definition 1.10: Dynamics

The characterization of all possible trajectories for all admissible ICs is called *dynamics* of a system.

1.2 Dimensional Analysis

Mathematics does not know units. Experimental physicists hate large sets of parameters because the sampling of high-dimensional parameter space is tiresome. A remedy to both issues is offered by the Buckingham-Pi-Theorem. We state it here in a form accessible with our present level of mathematical refinement. The discussion of a more advanced formulation may appear as a homework problem later on on this course.

Theorem 1.1: Buckingham-Pi-Theorem

A dynamics with n parameters, where the positions $\vec{q}$ and the parameters involve the three units meter, seconds and kilogram, can be rewritten in terms of a *dimensionless dynamics* with $n-3$ parameters, where the positions $\vec{\xi}$, velocities $\vec{\zeta}$, and parameters π_j with $j \in \{1, \ldots, n-1\}$ are given solely by numbers.

Example 1.7: Non-dimensionalization for a pendulum

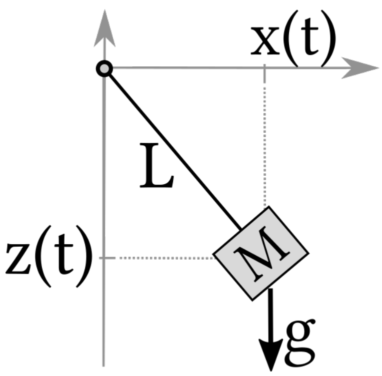


Figure 1.2: Pendulum discussed in Example 1.7

Let $\vec{x}$ denote the position of a pendulum of mass M that is attached to a chord of length L and swinging in a gravitational field $\vec{g}$ of strength g (see Figure 1.2).

The units of these quantities are $[\vec{x}] = \mathrm{m}$, $[M] = \mathrm{kg}$, $L = \mathrm{m}$, and $[g] = \mathrm{m/s^2}$, respectively. There are three parameters, $n = 3$, plus the direction of $\vec{g}$.

In this problem we choose L as length scale and $\sqrt{L/g}$ as time scale. Then the dimensionless positions will be $\vec{\xi} = \vec{q}/L$, the dimensionless velocities will be $\vec{\zeta} = \vec{\dot{q}}/\sqrt{g\,L}$.

There is no way to turn M into a dimensionless parameter. Therefore the evolution of (ξ, ζ) can not depend on M. The only dimensionless parameter that remains in the model is the direction of $\vec{g}$.

Example 1.8: Non-dimensionalization for the flight of a piece of chalk

Let $\vec{x}_1$ and $\vec{x}_2$ denote the position of the tip and the tail of a model for a piece of chalk, where tip and tail are associated to masses m_1 and m_2. The piece of chalk has a length L. It performs a free flight in a gravitational field with acceleration $\vec{g}$ of strength g.
The units of these quantities are $[\vec{x}_i] = \mathrm{m}$, $[m_i] = \mathrm{kg}$, $L = \mathrm{m}$, and $[g] = \mathrm{m/s^2}$, respectively. There are four parameters, $n = 4$, plus the direction of $\vec{g}$.
In this problem we choose L as length scale and $\sqrt{L/g}$ as time scale. Then the dimensionless positions will be $\vec{\xi} = \vec{q}/L$, the dimensionless velocities will be $\vec{\zeta} = \vec{q}/\sqrt{g\,L}$.
The two masses m_1 and m_2 give rise to the dimensionless parameter $\pi_1 = m_1/m_2$, and in three dimensions the direction of $\vec{g}$ must be characterized by another two dimensionless parameters.

Proof. We first look for combinations of the parameters with the following units

$$\mathrm{m} = \left[p_1^{\alpha_1}\right] \left[p_2^{\alpha_2}\right] \dots \left[p_n^{\alpha_n}\right]$$
$$\mathrm{s} = \left[p_1^{\beta_1}\right] \left[p_2^{\beta_2}\right] \dots \left[p_n^{\beta_n}\right]$$
$$\mathrm{kg} = \left[p_1^{\gamma_1}\right] \left[p_2^{\gamma_2}\right] \dots \left[p_n^{\gamma_n}\right]$$

Each of these equations involves constraints on the exponents in order to match the exponents of the three units that can be expressed as a system of linear equations. The solvability conditions for such systems imply that they conditions can always be met by an appropriately chosen set of three parameters. Without loss of generality we denote them as p_1, p_2 and p_3, and we have

$$\begin{aligned} \mathrm{m} &= \left[p_1^{\alpha_1}\right] \left[p_2^{\alpha_2}\right] \left[p_3^{\alpha_n}\right] \\ \mathrm{s} &= \left[p_1^{\beta_1}\right] \left[p_2^{\beta_2}\right] \left[p_3^{\beta_n}\right] \\ \mathrm{kg} &= \left[p_1^{\gamma_1}\right] \left[p_2^{\gamma_2}\right] \left[p_3^{\gamma_n}\right] \end{aligned} \tag{1.2.1}$$

Thus we use the parameters $p_1, \dots, p_3$ to remove the units from our description. In its dimensionless form it will involve the positions and velocities

$$\vec{\xi} = \vec{q}\, p_1^{-\alpha_1}\, p_2^{-\alpha_2}\, p_3^{-\alpha_n}$$
$$\vec{\zeta} = \dot{\vec{q}}\, p_1^{\beta_1-\alpha_1}\, p_2^{\beta_2-\alpha_2}\, p_3^{\beta_n-\alpha_n}$$

Similarly, the dimensionless form of the parameters p_i of the dynamics are obtained by multiplying the original parameters with appropriate powers of the expressions (1.2.1) of the units. For p_1 to p_3 this gives rise to one. Additional parameters will turn into dimensionless groups of parameters π_1 to π_{n-3}. □

1.3 *Order-of-magnitude guesses*

Many physical quantities take a value close to one when they are expressed in their "natural" dimensionless units. When the choice is unique, then clearly it is also natural. Otherwise, the appropriate choice is a matter of experience.

We will come back to this when we employ non-dimensionalization in the forthcoming discussion. We demonstrate this based on a discussion of

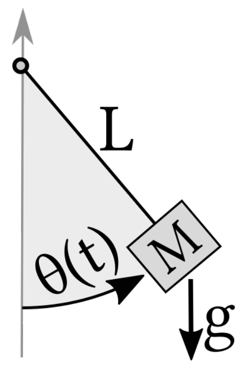


Figure 1.3: Pendulum discussed in Example 1.9

Example 1.9: The period of a pendulum

We consider a pendulum of mass M attached at a stiff bar of negligible mass. With this bar it is fixed to a pivot at a distance L from the mass such that it can swing in a gravitational field inducing an acceleration g (see Figure 1.3).
As discussed in Example 1.7 the dimensionless time unit for this problem is $\sqrt{L/g}$. Hence we estimate that the period T of the pendulum is $T \simeq \sqrt{L/g}$. Explicit calculations to be performed later on will reveal that this estimate is off by a factor 2π when the period is small. For large oscillation amplitudes θ_0 the period will increase further, tending to infinity when θ_0 approaches π. Hence, we conclude that

$$T = f(\theta_0)\sqrt{L/g} \qquad \text{with } \; f(\theta_0) \simeq 2\pi \; \text{ for } \; \theta_0 \ll 1\,.$$

Example 1.10: The speed of Tsunami waves

A Tsunami wave is a water wave that is generated by an earth quake or an underwater land slide. Typical wave lengths are of an order of magnitude $\lambda = 100\,\mathrm{km}$. They travel through the ocean that has an average depth of about $D = 4\,\mathrm{km}$, much smaller than λ. Therefore, we expect that the wave speed v_{Tsunami} is predominantly set by the ocean depth and the gravitational acceleration $g \approx 10\,\mathrm{m/s^2}$, i.e.

$$v_{\mathrm{Tsunami}} \approx \sqrt{gD} = 2 \times 10^2\,\mathrm{m/s} \approx 700\,\mathrm{km/h}$$

This estimate is very close to the observed values. Hence, the 2004 Indian Ocean Tsunami traversed the distance from Indonesia to the East African coast, $L \approx 10\,000\,\mathrm{km}$ in only

$$\frac{L}{v_{\mathrm{Tsunami}}} \approx \frac{1 \times 10^4\,\mathrm{km}}{700\,\mathrm{km/h}} = \frac{100}{7}\mathrm{h} \approx 15\,\mathrm{h}$$

It was 16 h according to Wikipedia. However, in spite of their speed and devastating power, Tsunamis are very hard to detect on the open sea because their period T is very long. It can be estimated as the time that the wave needs to run once through its wavelength[1]

$$T \approx \frac{\lambda}{v_{\mathrm{Tsunami}}} = \frac{\lambda}{\sqrt{gD}} = \frac{100\,\mathrm{km}}{700\,\mathrm{km/h}} = \frac{1}{7\,\mathrm{h}} \approx 10\,\mathrm{min}$$

Here, our estimate is too small by about a factor of three.

[1] Observe that this physical argument goes beyond the blind use of dimensional analysis. The equation for T involves the length scales λ and D in a non-trivial combination that is set by a physical argument.

We conclude that estimates based on dimensional analysis provide valuable insight in time scales of physical processes, even in situations where a detailed mathematical treatment is very delicate.

1.4 Problems

Rehearsing Concepts

Problem 1.1. Printing the output of Phantom cameras
With a set of three phantom cameras one can simultaneously follow the motion of 100 particles in a violent 3d turbulent flow. Data analysis of the images provides particle positions with a resolution of 25,000 frames per second. You follow the evolution for 20 minute, print it double paged with 8 coordinates per line and 70 lines per

page. A bookbinder makes 12 cm thick books from every 1000 pages. You put these books into bookshelves with seven boards in each shelf. How many meters of bookshelves will you need to store your data on paper?

Practicing Concepts

Problem 1.2. Oscillation Period of a Particle attached to a spring
In a gravitational field with acceleration $g_{\text{Moon}} = 1.6\,\text{m/s}^2$ a particle of mass $M = 100\,\text{g}$ is hanging at a spring with spring constant $k = 1.6\,\text{kg/s}^2$. It oscillates with period T when it is slightly pulled downwards and released. We describe the oscillation by the distance $x(t)$ from its rest position.

a) Determine the dimensionless distance $\xi(t)$,
and the associated dimensionless velocity $\zeta(t)$.

b) Provide an order-of-estimate guess of the oscillation period T.

Problem 1.3. Earth orbit around the sun

a) Light travels with a speed of $c \approx 3 \times 10^8\,\text{m}\,\text{s}^{-1}$. It takes 8 minutes and 19 seconds to travel from Sun to Earth. What is the distance D of Earth and Sun in meters?

b) The period of the trajectory of the Earth around the Sun depends on D, on the mass $M = 2 \times 10^{30}\,\text{kg}$ of the sun, and on the gravitational constant $G = 6.7 \times 10^{-11}\,\text{m}^3/\text{kg}\,\text{s}^2$. Estimate, based on this information, how long it takes for the Earth to travel once around the sun.

c) Express your estimate in terms of years. The estimate of (b) is of order one, but still off by a considerable factor. Do you recognize the numerical value of this factor?

d) Upon discussing the trajectory $\vec{x}(t)$ of planets around the sun later on in this course, we will introduce dimensionless positions of the planets $\vec{\xi}(t) = \vec{x}(t)/L = (x_1(t)/L, x_2(t)/L, x_3(t)/L)$. How would you define the associated dimensionless velocities?

Problem 1.4. Water Waves
The speed of waves on the ocean depends only on their wave length L and the gravitational acceleration $g \simeq 10\,\text{m/s}^2$.

a) How does the speed of the waves depend on L and g?

b) Unless it is surfing, the speed of a yacht is limited by its hull speed, i.e. the speed of a wave with wave length identical to the length of the yacht. Estimate the top speed of a 30 ft yacht.

c) Close to the beach the water depth H become a more important parameter than the wave length. How does the speed of the crest and the trough of the wave differ? What does this imply about the form of the wave?

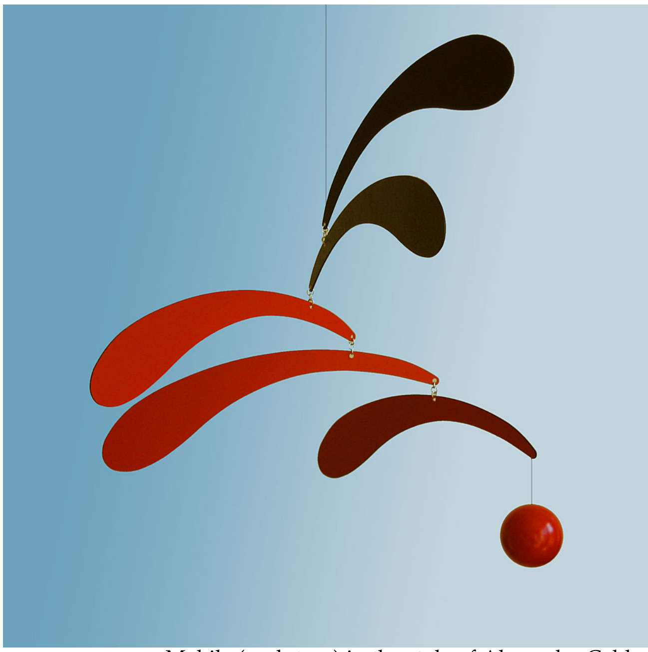
Proofs

Transfer and Bonus Problems, Riddles

2

Balancing forces and torques

In Chapter 1 we oberserved that positions and velocities of particles are specified by indicating their unit, magnitude and directions. Hence, they are vectors. In the present chapter we learn how vectors are defined in mathematics, and how they are used and handled in physics. In order to provide a formal definition we will introduce a number of mathematical concopts, like groups, that will be revisited in forthcoming chapters. As first important application we will deal with balancing forces and torques.



Mobile (sculpture) in the style of Alexander Calder
[wikimedia Creative Commons Attribution-Share Alike 2.0 Generic]

At the end of this chapter we will be able to determine how a mobile will be hanging from the ceiling.

2.1 Motivation and Outline: What is a vector?

In mechanics we use vectors to describe forces, displacements and velocities. A displacement describes the relative position of two points in space, and the velocity can be thought of as a distance divided by the time needed to go from the initial to the final point. (A mathematically more thorough definition will be given below.) For forces it is of paramount importance to indicate in which direction they are acting. Similarly, in contrast to speed, a velocity can not be specified in terms of a number with a unit, e.g. $5\,\mathrm{m\,s^{-1}}$. By its very definition one also has to specify the direction of motion. Finally, also a displacement involves a length specification and a direction. For instance, the displacement from the lower left corner of a piece of paper to a point in the middle can either be specified in terms if the distance r of the point from the corner and the angle θ of the line connecting the points and the lower edge of the paper (i.e. the direction of the point). Alternatively, it can be given in terms of two distances (x, y) that refer to the length x of a displacement along the edge of the paper and a displacement y in the direction vertical to the edge towards the paper. In three dimensions, one has to adopt a third direction out of the plane used for the paper, and hence three numbers, to specify a displacements—or indeed any other vector.

	displacement $\vec{x} = (x_1, x_2, x_3)$	velocity $\vec{v} = (v_1, v_2, v_3)$	force $\vec{F} = (f_1, f_2, f_3)$
unit	$[\vec{x}] = \mathrm{m}$	$[\vec{v}] = \mathrm{m\,s^{-1}}$	$[\vec{F}] = \mathrm{kg\ m/s^2}$
magnitude	$\lvert\vec{x}\rvert = \sqrt{x_1^2 + x_2^2 + x_3^2}$	$\lvert\vec{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	$\lvert\vec{F}\rvert = \sqrt{f_1^2 + f_2^2 + f_3^2}$
direction	$\hat{x} = \vec{x}/\lvert\vec{x}\rvert$	$\hat{v} = \vec{v}/\lvert\vec{v}\rvert$	$\hat{F} = \vec{F}/\lvert\vec{F}\rvert$

A basic introduction of mechanics can be given based on this heuristic accout of vectors. However, for the thorough exposition that serve as a fundament of theoretical physics a more profound mathematical understanding of vectors is crucial. Hence, a large part of this chapter will be devoted to mathematical concepts.

Outline

In the first part of this chapter we will introduce the mathematical notions of sets and groups that are needed to provide a mathematically sound definition of a vector space. Sets are the most fundamental structure of mathematics. It denotes a collection of elements, e.g., numbers like the digits of our number system $\{1, 2, \ldots, 9\}$ or the set of students in my class. Mathematical structures refer to sets where the elements obey certain additional properties, like in groups and vector spaces. They are expressed in terms of *operations* that take one or several elements of the set, and return a result that may or may not be part of the given set. When an operation f takes an element of a set A and returns another element of A we write $f : A \to A$. When an operation $\circ$ takes two elements of a set A and returns a single element of A we write $\circ : A \times A \to A$. Equipped with the mathematical tool of vectors we will explore the physical concepts of forces and torques, and how they are balanced in objects that do not move.

2.2 Sets

In mathematics and physics we often wish to make statements about a collection of objects, numbers, or other distinct entities.

Definition 2.1: Set

A *set* is a gathereing of well-defined, distinct objects of our perception or thoughts.
An object a that is part of a set A is an *element* of A; we write $a \in A$.
If a set M has a finite number n of elements we say that its *cardinality* is n. We write $|M| = n$.

Remark 2.1. Notationens and additional properties:

a) When a set M has a finite number of elements, e.g., $+1$ and -1, one can specify the elements by explicitly stating the elements, $M = \{+1, -1\}$. The order does not play a role and it does not make a differnece when elements are provided sevaral times. In other words the set M of cardinality two can be specified by any of the following statements

$$M = \{-1, +1\} = \{+1, -1\} = \{-1, 1, 1, 1, \} = \{-1, 1, +1, -1\}$$

b) If e is not an element of a set M, we write $e \notin M$. For instance $-1 \in M$ and $2 \notin M$.

c) There is only one set with no elements, i.e., with cardinality zero. It is denoted as $\emptyset$.

Example 2.1: Sets

- Set of English names of month:
  $A_1 = \{$January, February, March, April, May, June, July, August, September, Oktober, November, December$\}$
- Set of capitals of German states:
  $A_2 = \{$Berlin, Bremen, Hamburg, Stuttgart, Mainz, Wiesbaden, München, Magdeburg, Saarbrücken, Potsdam, Kiel, Hannover, Dresden, Schwerin, Düsseldorf, Erfurt$\}$
- Set of small letters in German:
  $A_3 = \{$a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z, ä, ö, ü, ß$\}$

The cardinalities of these sets are
$|A_1| = 12$, $|A_2| = 16$, and $|A_3 = 30|$.

Example 2.2: Sets of sets

A set can be an element of a set. For instance the set

$$M = \{1, 3, \{1, 2\}\}$$

has three elements 1, 3 und $\{1, 2\}$ such that $|M| = 3$, and

$$1 \in M\,, \qquad \{1, 2\} \in M\,, \qquad 2 \notin M \qquad \{1\} \notin M\,.$$

Often it is bulky to list all elements of a set. In obvious cases we use ellipses such as $A_3 = \{$a, b, c, $\dots$, z, ä, ö, ü, ß$\}$ for the set given in Example 2.1. Alternatively, one can provide a set M by specifying the properties of its elements x in the following form

$$\underbrace{M}_{\text{The set } M} \underbrace{=}_{\text{contains}} \underbrace{\{}_{\text{all elements}} \underbrace{x}_{x,} \underbrace{:}_{\text{with :}} \underbrace{A(x)}_{\text{properties} \dots} \}.$$

where the properties specify one of several properties of the elements. The properties are separated by commas, and must all be true for all elements of the set.

Example 2.3: Set definition by property

The set of digits $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ can also be defined as follows $D = \{x : 0 < x \leq 9, x \in \mathbb{Z}\}$.

In order to specify the poperties in a compact form we use logical junctors as short hand notation. In the present course we adopt the notations not $\neg$, and $\wedge$, or $\vee$, implies $\Rightarrow$, and is equivalent $\Leftrightarrow$ for the relations indicated in 2.1.

A	0	0	1	1	
B	0	1	0	1	
$\neg A$	1	1	0	0	not A
$\neg B$	1	0	1	0	not B
$A \vee B$	0	1	1	1	A or B
$A \wedge B$	0	0	0	1	A and B
$A \Rightarrow B$	1	1	0	1	A implies B
$A \Leftrightarrow B$	1	0	0	1	A is equivalent to B
$A \vee \neg B$	1	0	1	1	A or not B
$\neg A \wedge B$	0	1	0	0	not A or B
$A \wedge \neg B$	0	0	1	0	A and not B

Table 2.1: List of the results of different junctors acting on two statements A und B. Here 0 and 1 indicate that a statement is wrong or right, respectively. In the rightmost column we state the contents of the expression in the left colun in words. The final three lines provide examples of more complicated expressions.

The definition of the digits in Example 2.3 entails that all elements of D are also numbers in $\mathbb{Z}$: we say that D is a subset of $\mathbb{Z}$.

Definition 2.2: Subsets and Supersets

The set M_1 is a *subset* of M_2, if all ements of M_1 are also contained in M_2. We write[1] $M_1 \subseteq M_2$. We denote M_2 then as *superset* of M_1, writing $M_2 \supseteq M_1$.
The set M_1 is a *proper subset* of M_2 when at least one of its elements is not contained in M_2. In this case $|M_1| < |M_2|$ and we write $M_1 \subset M_2$, or $M_2 \supset M_1$.

[1] Some authors use $\subset$ instead of $\subseteq$, and $\subsetneq$ to denote proper subsets.

Example 2.4: Subsets

- The set of month with names the end with "'ber"' is a subset of the set A_2 of Example 2.1

$$\{\text{September, October, November, December}\} \subseteq A_3$$

- For the set M of Example 2.2 one has

$$\{1\} \subseteq M\,, \qquad \{1,3\} \subseteq M\,, \qquad \{1,2\} \not\subseteq M\,, \qquad \{2,\{1,2\}\} \not\subseteq M\,.$$

Note that $\{1, 2\}$ is an elements of M. However, it is not a subset. The last two sets are no subsets because $2 \notin M$.

Two sets are the same when they are subsets of each other.

Theorem 2.1: Equivalence of Sets

Two sets A and B are *equal* or *equivalent*, if $(A \subseteq B) \wedge (B \subseteq A)$.

Proof. $A \subseteq B$ implies that $a \in A \Rightarrow a \in B$.
$B \subseteq A$ implies $b \in B \Rightarrow b \in A$.
If $A \subseteq B$ and $B \subseteq A$, then we also have $a \in A \Leftrightarrow a \in B$. □

The description of sets by properties of its members, Example 2.3, suggests that one will often be interested in operations on sets. For instance the odd and even numbers are subsets of the natural numbers, together the form this set, and when one removed the odd numbers from the natural numbers one is left with the even numbers. Hence, we define the following operations on sets.

Definition 2.3: Set Operations

For two sets M_1 and M_2 we define the following operations:

- *Intersection*: $M_1 \bigcap M_2 = \{m \, : \, m \in M_1 \wedge m \in M_2\}$,
- *Union*: $M_1 \bigcup M_2 = \{m \, : \, m \in M_1 \vee m \in M_2\}$,
- *Difference*: $M_1 \backslash M_2 = \{m \, : \, m \in M_1 \wedge m \notin M_2\}$,
- The *complement* of a set M in a *universe* U is defined for subsets $M \subseteq U$ as follows $M^C = \{m \in U \, : \, m \notin M\}$.
- The *cartesian product of two sets* M_1 *and* M_2 is definined as the set of ordered pairs (a, b) of elements $a \in M_1$ and $b \in M_2$,
$$M_1 \times M_2 = \{(a, b) \, : \, a \in M_1, b \in M_2\} \ .$$

In the logical expressions following : the $\in$ and $\notin$ are evaluated with higher priority as the junktors $\wedge$ and $\vee$.

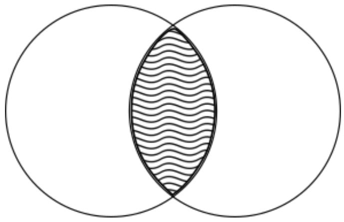


Figure 2.1: Intersection of two sets.

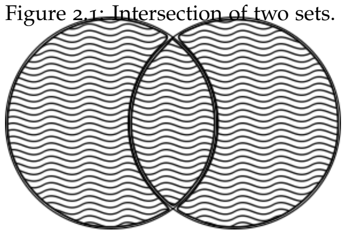


Figure 2.2: Union of two sets.

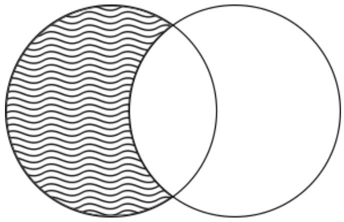


Figure 2.3: Difference of two sets.

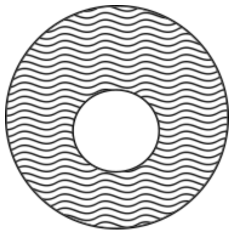


Figure 2.4: Complement of a set.

Example 2.5: Set operations for participants in my class

Consider the set of participants P in my class. The sets of female F and male M participants of the class are proper subsets of P with an empty intersection $F \bigcap M$. The set of non-female participants is $P \backslash W$. The set of heterosexual couples in the class is a subset of the caresian product $F \times M$. Furthermore, the union of the union of $W \bigcup M$ is a proper subset of P, when there is at least one participant who is neither female nor male.

Sets of Numbers

Many sets of numbers that are of interest in physics have infinite many elements. We construct them in Table 2.2 based on the natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

or the natural numbers with zero

$$\mathbb{N}_0 = \mathbb{N} \bigcup \{0\} \ .$$

Remark 2.2. Some authors adopt the convention that zero is in-

Table 2.2: Summary of important sets of numbers.

name	symbol	description
natural numbers	$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
natural numbers with 0	$\mathbb{N}_0$	$\mathbb{N} \bigcup \{0\}$
negative numbers	$-\mathbb{N}$	$\{-n \,:\, n \in \mathbb{N}\}$
even numbers	$2\mathbb{N}$	$\{2n \,:\, n \in \mathbb{N}\}$
odd numbers	$\mathbb{Z}$	$(-\mathbb{N}) \bigcup \mathbb{N}_0$
rational numbers	$\mathbb{Q}$	$\left\{\frac{p}{q} \,:\, p \in \mathbb{Z}, q \in \mathbb{N}\right\}$
real numbers	$\mathbb{R}$	see below
complex numbers	$\mathbb{C}$	$\mathbb{R} + \mathrm{i}\mathbb{R}$, wobei $\mathrm{i} = \sqrt{-1}$

cluded in the natural numbers $\mathbb{N}$. When this matters you have to check convention is adopted.

There are many more sets of numbers. For instance, in mathematics the set of *constructable numbers* is relevant for certain proors in geometry, and in physics we occasionally use *quaternions*. In any case one needs intervals of numbers.

Definition 2.4: Intervals of real numbers $\mathbb{R}$

An *interval* is a continuous subset of a set of numbers. We distinguish *open, closed, and half-open subsets.*

- closed interval: $[a, b] = \{x \,:\, x \geq a, x \leq b\}$,
- open interval: $(a, b) =]a, b[= \{x \,:\, x > a, x < b\}$,
- right open interval: $[a, b) = [a, b[= \{x \,:\, x \geq a, x > b\}$,
- left open interval: $(a, b[=]a, b] = \{x \,:\, x > a, x \geq b\}$.

Subsets of $\mathbb{R}$ will be denoted as real intervals.

2.3 Groups

A group refers to a set of operations that are changing some data or objects. elementary examples refer to the reflections in space, translations in space, or turning some sides of a Rubick's cube. The subsequent action of two group elements t_1 and t_2 will be considered to be another (typically more complicated) transformation t_3. Analogous to the concatenation of functions, we write $t_3 = t_2 \circ t_1$, and we say t_3 is t_2 after t_1. The set of transformations forms a group when one can always return to the starting point.

Definition 2.5: Group

A set $(G, \circ)$ is called a *group* with operation $\circ : G \times G \to G$ when the following rules apply

a) The set is *closed*: $\forall g_1, g_2 \in G : g_1 \circ g_2 \in G$

b) The set has a *neutral element*:
$$\exists e \in G \; \forall g \in G : e \circ g = g$$

c) Each element has an *inverse element*:
$$\forall g \in G \; \exists i \in G : g \circ i = e$$

d) The operation is *associative*:
$$\forall g_1, g_2, g_3 \in G : (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

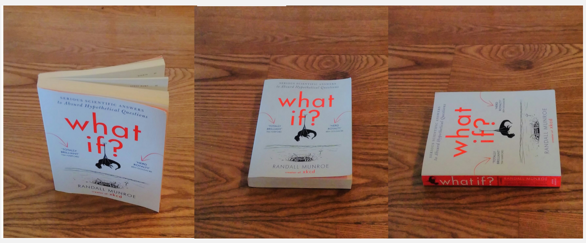
Definition 2.6: Kommutative Group

A group $(G, \circ)$ is called a *commutative group* when

e) The operation is *commutative*:
$$\forall g_1, g_2 \in G : g_1 \circ g_2 = g_2 \circ g_1$$

Example 2.6: Rotation-Groups are not commutative

The rotation of an object in space is a group. In particular this holds for the 90°-rotations of an object around a vertical and a horizontal axis. The figures bolow show that these rotations do not commute:



Example 2.7: Editing text fields

We condider the text fields of a fixed length n in an electronic form. Then the operations
"Put the letter $\square$ into postion $\bigcirc$ of the feild"
with $\square \in \{\_, a, \ldots, z, A, \ldots, Z\}$
and $\bigcirc \in \{1, \ldots, n\}$ form a group.
Also in this case one can easily check that the order of the operations is relevant. Verify that the same operations are preformed in the left and right column of the following graphics:

$$\begin{array}{rr}
|_|_|_|_| & |_|_|_|_| \\
\to |M|_|_|_| & \to |P|_|_|_| \\
\to |M|a|_|_| & \to |P|h|_|_| \\
\to |M|a|t|_| & \to |P|h|y|_| \\
\to |M|a|t|h| & \to |P|h|y|s| \\
\to |M|a|t|s| & \to |P|h|y|h| \\
\to |M|a|y|s| & \to |P|h|t|h| \\
\to |M|h|y|s| & \to |P|a|t|h| \\
\to |P|h|y|s| & \to |M|a|t|h|
\end{array}$$

Remark 2.3. Notationens and additional properties:

a) Depending of the context the inverse element will be denoted as g^{-1} oder als $-g$ notiert. This will depend on whether the operation is considered a multiplication or rather an addition. In accordance with this choice the netral element will be denoted as 1 or 0.

b) the requirement (b) $e \circ g = g$ implies that also $g \circ e = g$. The proof is provided as exercise **??**.

c) When a group is not abelian then one must distinguish the left and rigtht inverse. The condition $g \circ i = e$ does not imply $i \circ g = e$. However, there always will be another element $j \in G$ such that $j \circ g = e$.

The empty set can not be a group because it has no neutral element. Therefore the smallest groups has a single element

Example 2.8: The smallest group

$(\{n\}, \circ)$ comprises only the neutral element.

Sets of Numbers

Besides being of importance to characterize the action of discrete symmetry oprations like reflections or rotations by fixed angles, groups are also important for us because they admit further characterization of sets of numbers.

The natural numbers are not a group. For the addition they are lacking the neutral elements, and for adding and multiplications they are lacking inverse elements.

In contrast the group $(\mathbb{Z}, +)$ is a commutative group with infinitely many elements.

Example 2.9: The group $(\mathbb{Z}, +)$

The numbers $\mathbb{Z}$ with operation $+$ form a group. This is demonstrated here by checking the group axioms.

a) addition of any two numbers provides a number:

$\forall x, y \in \mathbb{Z} \,:\, (x + y) \in \mathbb{Z}.$

b) The neutral element of the addition is 0:

$\exists\, 0 \in \mathbb{Z}\ \forall z \in \mathbb{Z} \,:\, z + 0 = z = 0 + z.$

c) For every element $z \in \mathbb{Z}$ there is an inverse $(-z) \in \mathbb{Z}$:

$\forall z \in \mathbb{Z}\ \exists (-z) \in \mathbb{Z} \,:\, z + (-z) = 0 = (-z) + z.$

d) The addition of numbers is associative:

$\forall z_1, z_2, z_3 \in \mathbb{Z} \,:\, z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$

However, the numbers $\mathbb{Z}$ still lack inverse elements of the multiplication. The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ are are commutative groups for addition and multiplication (with the special rule that multiplication with 0 has no inverse element), and their elements also obey distributivity. Such sets are called number fields.

Definition 2.7: Field

A set $(\mathbb{F}, +, \cdot)$ is called a *field* with neutral elements 0 and 1 for addition $+$ and multiplication $\cdot$, respectively, when its elements comply with the following rules

a) $(\mathbb{F}, +)$ is a commutative group,

b) $(\mathbb{F}\backslash\{0\}, \cdot)$ is a commutative group,

c) Addition and Multiplication are distributive:
$$\forall a, b, c \in \mathbb{F} : a \cdot (b + c) = a \cdot b + a \cdot c$$

Remark 2.4. For the multiplication of field elements one commonly suppresses the $\cdot$ for the multiplication, writing e.g. $a\, b$ rather than $a \cdot b$.

2.4 *Vector Spaces*

With the notions introduced in the preciding sections we can give now the formal definition of a vector space

Definition 2.8: VectorSpace

A *vector space* $(\mathsf{V}, \mathbb{F}, \oplus, \odot)$ is a set of *vectors* $\vec{v}$ over a field $(\mathbb{F}, +, \cdot)$ with binary operations $\oplus : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ and $\odot : \mathbb{F} \times \mathsf{V} \to \mathsf{V}$ complying with the following rules

a) $(\mathsf{V}, \oplus)$ is a commutative group

b) associativity: $\forall a, b \in \mathbb{F}\ \forall \vec{v} \in \mathsf{V} : a \odot (b \odot \vec{v}) = (a \cdot b) \odot \vec{v}$

c) distributivity 1:
$$\forall a, b \in \mathbb{F}\ \forall \vec{v} \in \mathsf{V} : (a + b) \odot \vec{v} = (a \odot \vec{v}) \oplus (b \odot \vec{v})$$

d) distributivity 2:
$$\forall a \in \mathbb{F}\ \forall \vec{v}, \vec{w} \in \mathsf{V} : a \odot (\vec{v} \oplus \vec{w}) = (a \odot \vec{v}) \oplus (a \odot \vec{w})$$

Remark 2.5. It is common use $+$ and $\cdot$ instead of $\oplus$ and $\odot$, respectively, with the understanding that it is clear from the context in the equation whether the symbols refer to operations involving vectors or only numbers.

Moreover, similar to the agreement for the multiplication of numbers, one commonly drop the $\odot$ for the multiplication, writing e.g. $a\, \vec{v}$ rather than $a \cdot \vec{v}$.

Example 2.10: Vector spaces: diplacements in the plane

For displacemnts we define the operation $\oplus$ as concatenation of displacements, and $\odot$ as increasing the length of the displacement by a given factor without touching the direction.

add proof

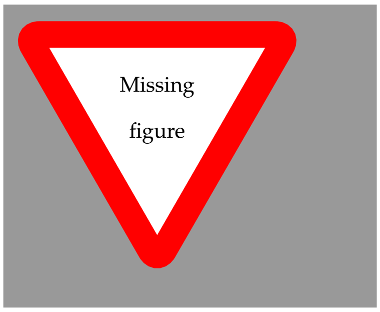


Figure 2.5: Graphical illustrations for Example 2.10

Example 2.11: Vector spaces: $\mathbb{R}^D$

For any $D \in \mathbb{N}$ the D-fold cartesian product $\mathbb{R}^D$ of the real numbers is a vector space over $\mathbb{R}$ when defining the operation $+$ and $\cdot$ as

$$\forall \vec{a}, \vec{b} \in \mathbb{R}^D : \vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_D \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_D \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \dots \\ a_D + b_D \end{pmatrix}$$

$$\forall s \in \mathbb{R}\ \forall \vec{a} \in \mathbb{R}^D : s \cdot \vec{a} = s \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_D \end{pmatrix} = \begin{pmatrix} s\, a_1 \\ s\, a_2 \\ \dots \\ s\, a_D \end{pmatrix}$$

The checking of the properties of a vector space is suggested as an exercise for the reader.

Example 2.12: Vector spaces: Polynomials of degree 2

For a field $\mathbb{F}$ the polynomials P of degree two in the variable x are defined as

$$\mathsf{P} = \{\vec{p} = (p_0 + p_1\,x + p_2\,x^2) : p_0, p_1, p_2 \in \mathbb{F}\}$$

This set is a vector space with respect to the summation

$$\begin{aligned}\vec{p} + \vec{q} &= (p_0 + p_1\,x + p_2\,x^2) + (q_0 + q_1\,x + q_2\,x^2)\\ &= ((p_0 + q_0) + (p_1 + q_1)\,x + (p_2 + q_2)\,x^2)\end{aligned}$$

and the multiplication with a scalar $s \in \mathbb{F}$

$$s \cdot \vec{p} = s \cdot (p_0 + p_1\,x + p_2\,x^2) = ((s\,p_0) + (s\,p_1)\,x + (s\,p_2)\,x^2)$$

add proof of vector-space properties

2.5 Forces

add “Forces”

2.6 Scalar Products and Coordinates

add “Scalar Products and Coordinates”

2.7 Torques and cross products

add “Torques and cross products”

2.8 The mobile — a worked example

add worked example “mobile”

2.9 Problems

Rehearsing Concepts

Problem 2.1. Checking group axioms
Which of the following sets are groups?

a) $(\mathbb{N}, +)$ c) $(\mathbb{Z}, \cdot)$ e) $(\{0\}, +)$

b) $(\mathbb{Z}, +)$ d) $(\{+1, -1\}, \cdot)$ $(\{1, \dots, 12\}, \oplus)$

where $\oplus$ in f) revers to adding as we do it on a clock, e.g. $10 \oplus 4 = 2$.

Problem 2.2. **Euler's equation and trigonometric relations**

Euler's equation $e^{ix} = \cos x + i \sin x$ relates complex values exponential functions and trigonometric functions.

a) Sketch the position of e^{ix} in the complex plain, and indicate how Euler's equation is related to the Theorem of Pythagoras.

b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular one has $e^{i(x+y)} = e^{ix} e^{iy}$. Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about $\sin(2x)$ and $\cos(2x)$?

Problem 2.3. **Carthesian Coordinates**

a) Mark the following points in a carthesian coordinate system:

$$(0;0) \quad (0;3) \quad (2;5) \quad (4;3) \quad (4;0)$$

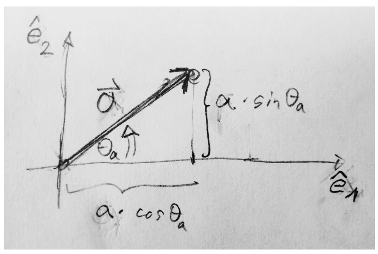
Add the points $(0;0)$ $(4;3)$ $(0;3)$ $(4;0)$, and connect the points in the given order. What do you see?

b) What do you find when drawing a line segment connecting the following points?

$$(0;0) \quad (1;4) \quad (2;0) \quad (-1;3) \quad (3;3) \quad (0;0)$$

Problem 2.4. **Geometric and algebraic form of the scalar product**

The sketch in the margin shows a vector $\vec{a}$ in the plane, and its representation as a linear combination of two orthonormal vectors $(\hat{e}_1, \hat{e}_2)$,



$$\vec{a} = a \cos\theta_a \, \hat{e}_1 + a \sin\theta_a \, \hat{e}_2$$

Here, a is the length of the vector $\vec{a}$, and $\theta_1 = \angle(\hat{e}_1, \vec{a})$.

a) Analogously to $\vec{a}$ we will consider another vector $\vec{b}$ with a representation

$$\vec{b} = b\, \cos\theta_b\, \hat{e}_1 + b\, \sin\theta_b\, \hat{e}_2$$

Employ the rules of scalar products, vector addition and multiplication with scalars to show that

$$\vec{a} \cdot \vec{b} = a\, b\, \cos(\theta_a - \theta_b)$$

Hint: Work backwards, expressing $\cos(\theta_a - \theta_b)$ in terms of $\cos\theta_a$, $\cos\theta_b$, $\sin\theta_a$, and $\sin\theta_b$.

b) As a shortcut to the explicit calculation of a) one can introduce the coordinates $a_1 = a\, \cos\theta_a$ and $a_2 = a\, \sin\theta_a$, and write $\vec{a}$ as a tuple of two numbers. Proceeding analogously for $\vec{b}$ one obtains

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

How will the product $\vec{a} \cdot \vec{b}$ look like in terms of these coordinates?

c) How do the arguments in a) and b) change for D dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors $\hat{e}_1, \dots, \hat{e}_D$?

What changes when the basis is not orthonormal?
What if it is not even orthogonal?

Problem 2.5. **Angles between three balanced forces**

Consider three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_p$ like in the rubber band example of the lecture, where I pull the band with force $\vec{F}_p$ and this force is balanced by the forces due to the tension of the rubber band.

a) Make a sketch of the setup where you indicate the angels $\angle(\vec{F}_p, \vec{F}_1)$ as θ_{1p} and $\angle(\vec{F}_p, \vec{F}_2)$ as θ_{2p}, respectively.

b) Determine the condition for a balance of forces in the directions parallel to $\vec{F}_p$ and parallel to $\vec{F}_1$.

c) The result of (b) can be expressed as a conditions on $F_p = |\vec{F}_p|$ as function of F_1, F_2, θ_{1p} and θ_{2p}, and on F_1 as function of F_p, F_2, θ_{1p} and θ_{2p}. Insert the former condition into the latter one in order to eliminate F_p.

Hence, you find that F_1 will be proportional to F_2, when the angles θ_{1p} and θ_{2p} are fixed. What does this reflect from a physical point of view?
Hint: What happens to the force balance when you fix the angle and increase $\vec{F}_p$ by a factor φ.

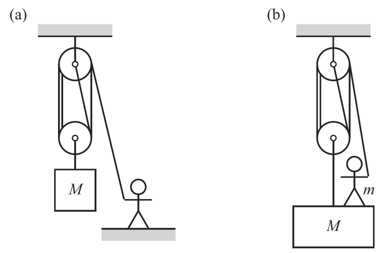
d) Employ trigonometric relations to show that the proportionality constant can be written as a ratio of two sines, i.e. one has

$$F_1 = \frac{\sin\alpha}{\sin\alpha} F_2$$

How are the angles α and β related to θ_{1p} and θ_{2p}?

e) Can you find a simpler way to derive the expression found in (d)?

Problem 2.6. **Tackling tackles and pulling pulleys**



a) Which forces are required to hold the balance in sketch (a) and (b)?

b) Let the sketched person and the weight have masses of 75 kg and 300 kg, respectively. Which power is required when to haul the line at a speed of 1 m/s.
Hint: The power is defined here as the change of (a) $Mg\,z(t)$ and (b) $(M+m)\,g\,z(t)$, per unit time, respectively.

Practicing Concepts

Problem 2.7. Properties of right-angled triangles

a) Fill in the gaps for the values of the angle θ in radians, and employ the symmetry of the trigonometric sine and cosine functions to determine the values in the right columns

θ		$\sin\theta$	$\cos\theta$
$\circ$	rad		
0	________	0	________
30	________	$\frac{1}{2}$	________
45	________	$\frac{\sqrt{2}}{2}$	________
60	________	$\frac{\sqrt{3}}{2}$	________
90	________	1	________
120	________	________	________
135	________	________	________
150	________	________	________
180	________	________	________

b) Consider a right triangle where one of the angles is θ. How are the length of its sides related to $\sin\theta$ and $\cos\theta$? Check that the Theorem of Pythagoras holds! Do you see a systematics for the angles?

c) Use the symmetries of the trigonometric functions to determine the values provided for $\theta = \pi/4$.

Use the symmetries of the trigonometric functions and the trigonometric relation for $\sin(2\theta)$ to determine the values provided for $\theta = \pi/6$ and $\theta = \pi/3$.

The values for $\pi/10$, $\pi/8$, and $\pi/5$ can also be stated explicitly in elementary form. Determine the expressions for these values!

Problem 2.8. **Linear Dependence of three vectors in 2D**

In the lecture I pointed out that every vector $\vec{v} = (v_1, v_2)$ of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors $\vec{a}$ and $\vec{b}$,

$$\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b}$$

In this exercise we revisit this statement for $\mathbb{R}^2$ with the standard forms of vector addition and multiplication by scalars.

a) Provide a triple of vectors $\vec{a}$, $\vec{b}$ and $\vec{v}$ such that $\vec{v}$ can *not* be represented as a scalar combination of $\vec{a}$ and $\vec{b}$.

b) To be specific we will henceforth fix

$$\vec{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,, \qquad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,, \qquad \vec{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers α and β such that

$$\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b}$$

c) Consider now also a third vector

$$\vec{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for (α, β, γ) such that

$$\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b} + \gamma\vec{c}$$

What is the general constraints on (α, β, γ) such that $\vec{v} = \alpha\,\vec{a} + \beta\,\vec{b} + \gamma\vec{c}$.
What does this imply on the number of solutions?

d) Discuss now the linear dependence of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ by exploring the solutions of

$$\vec{0} = \alpha\,\vec{a} + \beta\,\vec{b} + \gamma\vec{c}$$

How are the constraints for the null vector related to those obtained in part c)?

Problem 2.9. **Torques acting on a ladder**

The sketch in the margin shows the setup of a ladder leaning to the roof of a hut. The indicated angle from the downwards vertical

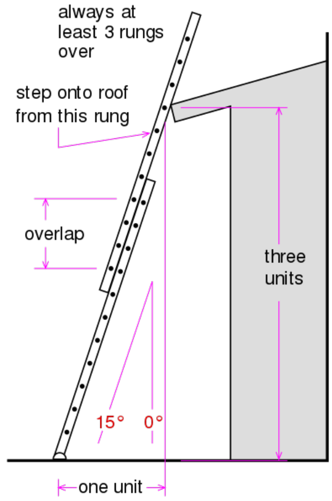


Figure 2.6: Sketch is due to Bradley and the vector image to Sarang [Public domain from wikimedia]

to the ladder will be denoted as θ. There is a gravitational force of magnitude Mg acting of a ladder of mass M. At the point where it leans to the roof there is a normal force of magnitude F_r acting from the roof to the ladder. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude γF_f. This is again the sketch to the ladder leaning to the roof of a hut. The angle from the downwards vertical to the ladder is denoted θ. There is a gravitational force of magnitude Mg acting of a ladder. At the point where it leans to the roof there is a normal force of magnitude F_r. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude F_f.

a) In principle there also is a friction force $\gamma_r F_r$ acting at the contact from the ladder to the roof. Why is it admissble to neglect this force?
Remark: There are at least two good arguments.

b) Determine the vertical and horizontal force balance for the ladder. Is there a unique solution?

c) The feet of the ladder start sliding when F_f exceeds the maximum static friction force γF_g. What does this condition entail for the angle θ?
Assume that $\gamma \simeq 0.3$ What does this imply for the critical angle θ_c.

d) Where does the mass of the ladder enter the discussion? Do you see why?

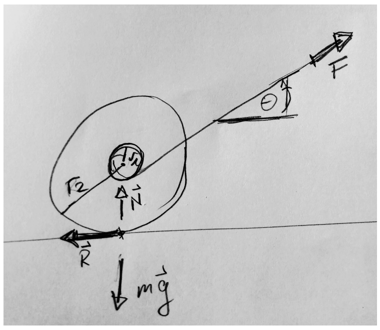
e) Determine the torque acting on the ladder. Does it matter whether you consider the torque with respect to the contact point to the roof, the center of mass, or the foot of the ladder?

f) The ladder will slide when the modulus of the friction force F_f exceeds a maximum value $\mu_S F_g$ where μ is the static friction coefficient for of the ladder feet on the ground. For metal feet on a wooden ground it takes a value of $\mu_S \simeq 2$. What does that tell about the angels wher the ladder starts to slide?

g) Why does a ladder commonly starts sliding when when a man has climbed to the top? Is there anything one can do against it? Is that even true, or just an urban legend?

Problem 2.10. **Walking a yoyo**

The sketch to the right shows a yoyo of mass m standing on the ground. It is held at a chord that extends to the top right. There are four forces acting on the yoyo: gravity $m\vec{g}$, a normal force $\vec{N}$ from the ground, a friction force $\vec{R}$ at the contact to the ground, and the force $\vec{F}$ due to the chord. The chord is wrapped around an axle of radius r_1. The outer radius of the yoyo is r_2.



a) Which conditions must hold such that there is no net force acting on the center of mass of the yoyo?

b) For which angle θ will the torque vanish?

c) Perform an experiment: What happens for larger and for smaller angels θ? How does the yoyo respond when fix the height where you keep the chord and pull continuously?

Proofs

Problem 2.11. **Different basis for polynomials**

We consider the set of polynomials $\mathbb{P}_N$ of degree N with real coefficients p_n, $n \in \{0, \dots, N\}$,

$$\mathbb{P}_N := \left\{ \vec{p} = \left(\sum_{k=0}^{N} p_n \, x^k \right) \quad \text{mit} \;\; p_n \in \mathbb{R}, n \in \{0, \dots, N\} \right\}$$

a) Demonstrate that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is a vector space when one adopts the operations

$$\forall \quad \vec{p} = \left(\sum_{k=0}^{N} p_n \, x^k \right) \in \mathbb{P}_N \,, \quad \vec{q} = \left(\sum_{k=0}^{N} q_n \, x^k \right) \in \mathbb{P}_N \,, \text{ and } \; c \in \mathbb{R} \;:$$

$$\vec{p} + \vec{q} = \left(\sum_{k=0}^{N} (p_k + q_k) \, x^k \right) \quad \text{and} \quad c \cdot \vec{p} = \left(\sum_{k=0}^{N} (c \, p_k) \, x^k \right) .$$

(b) Demonstrate that

$$\vec{p} \cdot \vec{q} = \left(\int_0^1 \mathrm{d}x \left(\sum_{k=0}^{N} p_k \, x^k \right) \left(\sum_{j=0}^{N} q_j \, x^j \right) \right) ,$$

establishes a scalar product on this vector space.

(c) Demonstrate that the three polynomials $\vec{b}_0 = (1)$, $\vec{b}_1 = (x)$ und $\vec{b}_2 = (x^2)$ form a basis of the vector space $\mathbb{P}_2$: For each

polynomial $\vec{p}$ aus $\mathbb{P}_2$ there are real numbers x_k, $k \in \{0,1,2\}$, such that $\vec{p} = x_0\,\vec{b}_0 + x_1\,\vec{b}_1 + x_2\,\vec{b}_2$. However, in general we have $x_i \neq \vec{p}\cdot\vec{b}_i$. Why is that?
Hint: Is this an orthonormal basis?

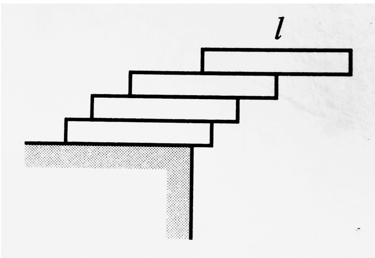
(d) Demonstrate that the three vectors $\hat{e}_0 = (1)$, $\hat{e}_1 = \sqrt{3}\,(2\,x-1)$ and $\hat{e}_2 = \sqrt{5}\,(6\,x^2 - 6\,x + 1)$ are orthonormal.

(e) Demonstrate that every vector $\vec{p} \in \mathbb{P}_2$ can be written as a scalar combination of $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$,

$$\vec{p} = (\vec{p}\cdot\hat{e}_0)\,\hat{e}_0 + (\vec{p}\cdot\hat{e}_1)\,\hat{e}_1 + (\vec{p}\cdot\hat{e}_2)\,\hat{e}_2\,.$$

Hence, $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$ form an orthonormal basis of $\mathbb{P}_2$.

*(f) Find a constant c and a vector $\hat{n}_1$, such that $\hat{n}_0 = (c\,x)$ and $\hat{n}_1$ form an orthonormal basis of $\mathbb{P}_1$.



Problem 2.12. **Piling bricks**

Christmas is approaching, and Germans consume enormous amounts of chocolate. If you happen to come across a considerable pile of chocolate bars (or beer mats, or books, or anything else of that form) I recommend the following experiment:

a) We consider N bars of length l piled on a table. What is the maximum amount that the topmost bar can reach beyond the edge of the table.

b) The sketch above shows the special case $N = 4$.
However, what about the limit $N \to \infty$?

Transfer and Bonus Problems, Riddles

Problem 2.13. **Systems of linear equations**

A system of N linear equations of M variables $x_1, \ldots x_M$ comprises N equations of the form

$$\begin{aligned}
b_1 &= a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1M}\,x_M \\
b_2 &= a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2M}\,x_M \\
\vdots \quad & \quad \vdots \\
b_N &= a_{N1}\,x_1 + a_{N2}\,x_2 + \cdots + a_{NM}\,x_M
\end{aligned}$$

where $b_i, a_{ij} \in \mathbb{R}$ for $i \in \{1,\ldots,N\}$ and $j \in \{1,\ldots,M\}$.

a) Demonstrate that the linear equations $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$ form a vector space when one adopts the operations

$$
\begin{aligned}
\forall \quad & \vec{p} = \left[p_0 = p_1\, x_1 + p_2\, x_2 + \cdots + p_M\, x_M\right] \in \mathbb{L}_N\,, \\
& \vec{q} = \left[q_0 = q_1\, x_1 + q_2\, x_2 + \cdots + q_M\, x_M\right] \in \mathbb{L}_N\,, \\
& c \in \mathbb{R} \;:
\end{aligned}
$$

$$
\begin{aligned}
\vec{p} + \vec{q} &= \left[p_0 + q_0 = (p_1 + q_1)\, x_1 + (p_2 + q_2)\, x_2 + \cdots + (p_M + q_M)\, x_M\right] \\
c \cdot \vec{p} &= \left[c\, p_0 = c\, p_1\, x_1 + c\, p_2\, x_2 + \cdots + c\, p_M\, x_M\right].
\end{aligned}
$$

How do these operations relate to the operations performed in Gauss elimination to solve the system of linear equations?

b) The system of linear equations can also be stated in the following form

$$
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M
$$

$$
\Leftrightarrow \qquad \vec{b} = x_1\, \vec{a}_1 + x_2\, \vec{a}_2 + \cdots + x_M\, \vec{a}_M
$$

where $\vec{b}$ is expressed as a linear combination of $\vec{a}_1, \ldots \vec{a}_M$ by means of the numbers $x_1, \ldots, x_M$. What do the conditions on linear independence and representation of vectors by means of a basis tell about the existence and uniqueness of the solutions of a system of linear equations.

Problem 2.14. **Eagle and Hedgehog**

a) You are looking from South into a valley and see a hedgehog that sits 10 m to the right of a tree on in the grass. From far right above an eagle is attacking. You monitor its position based on coordinates to the right and vertical from the foot of the tree, indicating distances in meter:

$$
(10.9\,;\,28) \quad (11.5\,;\,44) \quad (12.1\,;\,62) \quad (13.2\,;\,99)
$$

Will the eagle catch the hedgehog?

b) A friend is taking a movie of the same incidence, observing it from the East. From the movie you extract the position of the eagle to the North and vertical with respect ot the foot of the tree.

$$(1.9\,;27)\quad(2.4\,;44)\quad(3.1\,;62)\quad(3.8\,;85)$$

Will the eagle catch the hedgehog?

c) A forester, who knows the animals really well, is telling you that the hedgehog will typically start moving North with a speed of about $1\,\mathrm{m\,s^{-1}}$ when he notices the eagle coming down. Will the eagle catch the hedgehog when it is coming down with a speed of $50\,\mathrm{m\,s^{-1}}$ and when the hedgehog notices the eagle at a heigt of $50\,\mathrm{m}$?

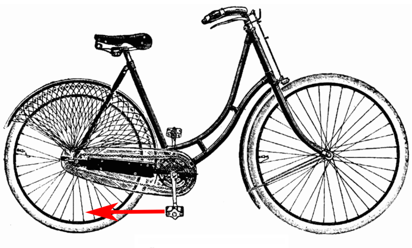


Figure 2.7: The picture of the bicycle is due to Otto Lueger, Damenfahrrad 1904 [Public domain, wikimedia].

Problem 2.15. **Where will the bike go?**

Consider the picture of the bicycle to the left. The red arrow indicates a force that is acting on the paddle in backward direction.

Will the bicycle move forwards or backwards?

Take a bike and do the experiment!

3

Newton's Laws

In Chapter 2 we explored how several forces that act on a body can be subsumed into a net total force and torque. The body stays in rest, say at position $\vec{q}_0$, when the net force and torque vanish. Now we explore how the forces induce motion and how the position of the body evolves in time, $\vec{q}(t)$, when it is prepared with an initial condition $\vec{q}(t_0) = \vec{q}_0$ at the initial time t_0.

At the end of this chapter we will be able to discuss the likelyhood for injuries in different types of accidents, be it men or cat or mice. Why do the cats ge away unharmed in most cases when they fall from a balcony, while an old professor should definitely avoid such a fall.

3.1 Motivation and Outline: What is causing motion?

Every now and then I make the experience that I sit in a train, reading a book. Then I look out of the window, realize that we are passing a train, feeling happy that we are further approaching my final destination; and then I realize that the train is moving and my train is still in the station. Indeed, the motion of objects in my compartment is exactly identical, no matter whether it is at rest or moves with a constant velocity; be it zero in the station, at $15\,\mathrm{m/s}$ in a local commuter train, or $75\,\mathrm{m/s}$ in a Japanese high-speed train. However, changes of speed matter. I forcefully experience the force exerted on the train during an emergency break.

Modern physics was born when Galilei and Newton formalized this experience by saying that bodies (e.g. the set of bodies in the compartment of a train) move in a straight line with a constant velocity as long as there is no net force acting on the bodies. A pulley pushed by a mine worker is moving at constant speed because the force induced by the mine worker is exactly balanced by friction forces. All that physics has to say about the cause of motion is that it is due to some force that has been working in the past. It speed and direction changes if and only if a force is acting.

Outline

In the first part of this chapter we will relate temporal changes of positions and velocities to time derivatives. Subsequently, we can formulate equations of motion that relate these changes to forces. To this end we will have to introduce the mathematical concept of differential equations. The last part of the chapter deals with strategies to find solutions and characterize sets of solutions of differential equations.

mass	m
position	$\vec{q}(t)$
velocity	$\dot{\vec{q}}(t), \vec{v}(t)$
acceleration	$\ddot{\vec{q}}(t)$
forces	$\vec{F}_\alpha(\vec{q}, t)$

Table 3.1: Notations adopted to describe the motion of a particle. A single dot denotes the time derivative, and double dot the second derivative with respect to time.

3.2 *Time derivatives of vectors*

In this section we consider the motion of a particle with mass m that is at position $\vec{q}(t)$ at time t. When it moves, then its average velocity $\vec{v}_{\text{av}}(t, \Delta t)$ during the time interval $[t, t + \Delta t]$ is

$$\vec{v}_{\text{av}}(t, \Delta t) = \frac{\vec{q}(t + \Delta t) - \vec{q}(t)}{\Delta t}$$

When the limit $\lim_{\Delta t \to 0} \vec{v}_{\text{av}}(t, \Delta t)$ exists[1] we can define the velocity of the particle at time t,

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{q}(t + \Delta t) - \vec{q}(t)}{\Delta t} \tag{3.2.1}$$

The velocity is then the time derivative of the position, and in an immediate generalization of the time derivative of scalar functions we also write

$$\dot{\vec{q}}(t) = \vec{v}(t) = \frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t}$$

Finally, we point out that the components of the time derivative of a vector amount to the derivatives of the components.

[1] The discussion of this limit of general functions is a core topic of vector calculus. For our present purpose the intuitive understanding based on the idea that $\vec{q}(t + \Delta t) \simeq \vec{q}(t) + \Delta t\ \vec{v}(t)$ provides the right idea. To provide a hint for the origin of the mathematical subtleties we point out that the approximation works unless there is an *instantaneous* collision with a wall at some point in the time interal $]t, t + \Delta t[$. In physics we try out luck, and fix the problem when we face it. Indeed, upon a close look there are no instantaneous collisions in physics.

Theorem 3.1: Time derivatives of vectors

Let $\vec{a}(t)$ be a vector with time-dependent components $a_i(t)$ with repect to orthonormal basis $\{\vec{e}_i, i = 1 \cdots D\}$ that is fixed in time.
Then $\dot{\vec{a}}(t) = \sum_i \dot{a}_i(t)\, \hat{e}_i$, i.e. the components of $\dot{\vec{a}}(t)$ amount to the time derivates of the components of $\vec{a}(t)$.

Proof. For each time we have $\vec{a}(t) = \sum_i a_i(t)\, \hat{e}_i$ where it is understood that the sum runs over $i = 1 \cdots D$. We insert this into the definition, Equation (3.2.1), of the the time derivative and use the linearity of scalar products with vectors to obtain

$$\begin{aligned}
\dot{\vec{a}}(t) &= \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sum_i a_i(t+\Delta t)\, \hat{e}_i - \sum_i a_i(t)\, \hat{e}_i}{\Delta t} \\
&= \lim_{\Delta t \to 0} \sum_i \hat{e}_i \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} = \sum_i \hat{e}_i \lim_{\Delta t \to 0} \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} \\
&= \sum_i \hat{e}_i\, \dot{a}_i(t)
\end{aligned}$$

The subtle step here, from a mathematical point of view, is the swapping of the limit and the sum in the second line of the argument. Courses on vector calculus will spell out the assumptions needed to justify this step (or, more interestingly from a physcis perspective, under which conditions it fails). □

The change of the velocity will be denoted as acceleration. Based on an analogous argument as for the velocity, it will be written as a time derivative

Definition 3.1: Acceleration

The time derivative of the velocity $\vec{v}(t) = \dot{\vec{q}}(t)$ will be denoted as acceleration, and written as

$$\frac{\mathrm{d}\vec{v}(t)}{\mathrm{d}t} = \dot{\vec{v}}(t) = \ddot{\vec{q}}(t)$$

In the next section it will be related to the action of forces $\vec{F}(\vec{q}, t)$ acting on a particle that resides at the position $\vec{q}$ at time t.

3.3 Newton's Axioms

In Section 4.1 we referred to a train compartment to point out that physical observations will be the same — irrespective of the velocity of its motion, as long as it is constant. As setting where we perform an experiment is denoted as reference frame, and reference frames that move with constant velocity are called intertial systems.

Definition 3.2: Reference Frames and Inertial Systems

A *reference frame* $(\vec{Q}, \{\hat{e}_i(t), i = 1 \cdots D\})$ is an agreement about the, in general time dependent, position of the origin $\vec{Q}(t)$ of the coordinate system and a set of orthonormal basis vectors $\{\hat{e}_i(t), i = 1 \cdots D\}$, that are adopted to indicate the positions of particles in a physical model.

The reference frame refers to an *inertial system* when it does not rotate and when it moves with a constant velocity, i.e., if and only if $\ddot{\vec{Q}} = \vec{0}$ and $\dot{\hat{e}}_i = \vec{0}$ for all $i \in \{1 \cdots D\}$.

1st Law

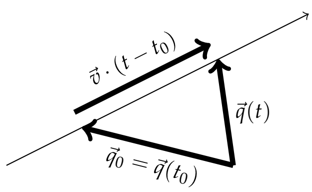
As long as a reference frame moves with a constant velocity, it feels like at rest. Physical measurements can only detect acceleration. This is expressed by

Axiom 3.1: Newton's 1st law

The velocity of a particle moving in an inertial system is constant, unless a (net) force is acting on the particle,

$$\begin{aligned} \forall t \geq t_0 : \vec{F}(t) = \vec{0} \quad &\Leftrightarrow \quad \dot{\vec{q}}(t) = \vec{v} = \text{const} \\ &\Leftrightarrow \quad \vec{q}(t) = \vec{q}_0 + \vec{v}\,(t - t_0) \end{aligned}$$

as sketched in the margin.



The particle moves then in a straight line with a constant speed. Indeed, when a particle moves with the constant velocity $\vec{v} = \dot{\vec{q}}(t)$ in the reference frame $(\vec{Q}, \{\hat{e}_i(t), i = 1 \cdots D\}$ then it is at rest in the alternative reference frame $(\vec{Q} + \vec{v}\,t, \{\hat{e}_i(t), i = 1 \cdots D\}$. Therefore, in the latter coordinate system the particle is at rest, and it will remain at rest when it is not perturbed by a net external force.

Proof. We choose the origin for the frame of reference to describe the position of the particle to be

$$\vec{Q} = \vec{q}(t)$$

By construction the particle is always in the origin of this coordinate frame, i.e., at rest, and for $\vec{q}(t) = \vec{q}_0 + \vec{v}\,(t - t_0)$ the coordinate frame is an inertial frame because

$$\dot{\vec{Q}} = \dot{\vec{q}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{q}_0 - \vec{v}\, t_0) + \dot{\vec{v}}\, t + \vec{v} = \vec{v} = \text{const}$$

where the first two terms vanish because $\vec{q}_0$, $\vec{v}$, and t_0 are constant.

□

2nd Law

Newton's second law spells out how the velocity of the particle changes when there is a force.

Axiom 3.2: Newton's 2nd law

The change, $\ddot{\vec{q}}(t)$, of the velocity of a particle, $\dot{\vec{q}}(t)$, at position, $\vec{q}(t)$, is proportional to the sum of acting forces $\vec{F}_\alpha$ with a propotionality factor m,

$$\ddot{\vec{q}}(t) = m \sum_\alpha \vec{F}_\alpha(t)$$

Remark 3.1. In general the time dependence of the forces can be decomposed into three contributions

a) An implicit time dependence, $\vec{F}(\vec{q}(t))$, when the force depends on the position, $\vec{q}(t)$ of the particle. For instance, for a Hookian spring with spring constant k one has, $\vec{F}(\vec{q}) = -k\,\vec{q}$

b) An implicit time dependence, $\vec{F}(\dot{\vec{q}}(t))$, when the force depends on the velocity, $\dot{\vec{q}}(t)$ of the particle. For instance, the sliding friction for a particle with mass m and friction coefficient γ is, $\vec{F}(\dot{\vec{q}}) = -m\,\gamma\,\dot{\vec{q}}$

c) An explicit time dependence when the force is changing in time. For instance, when pushing a child sitting on a swing one will only push when the swing is moving in forward direction.

Typically, one explicitly sorts out these dependences and writes

$$\ddot{\vec{q}}(t) = m \sum_\alpha \vec{F}_\alpha(\vec{q}(t), \dot{\vec{q}}(t), t)$$

Example 3.1: Particle moving in the gravitational field

The gravitational field induces a constant force $m\,\vec{g}$ on a particle with mass m. Let it have velocity $\vec{v}_0$ at time t_0 when it is taking off from the position $\vec{q}_0$. Then Newton's 2nd law states that $\ddot{q}(t) = \vec{g}$, and this equation must be solved subject to the initial conditions $\vec{q}(t_0) = \vec{q}_0$ and $\dot{\vec{q}}(t_0) = \vec{v}$. By working out the derivatives one readily checks that this is given for

$$\vec{q}(t) = \vec{q}_0 + \vec{v}\,(t - t_0) + \frac{1}{2}\,\vec{g}\,(t - t_0)^2$$

Example 3.2: Particle moving in a circle

Let a particle of mass m move with constant speed in a circle of radius R such that its position can be written as

$$\vec{q}(t) = \begin{pmatrix} R\ \cos(\omega t) \\ R\ \sin(\omega t) \end{pmatrix}$$

with a constant angular velocity ω. Then its velocity and acceleration take the form

$$\begin{pmatrix} -\omega\,R\ \sin(\omega t) \\ \omega\,R\ \cos(\omega t) \end{pmatrix} \quad \text{and} \quad \ddot{\vec{q}}(t) = \begin{pmatrix} -\omega^2\,R\ \cos(\omega t) \\ -\omega^2\,R\ \sin(\omega t) \end{pmatrix} = -\omega^2\,R\,\vec{q}(t)$$

The speed is constant, taking the value $\sqrt{\dot{\vec{q}} \cdot \dot{\vec{q}}} = \omega\,R$ and the force is antiparallel to $\vec{q}$ with magnitude $m\,\omega^2\,R$. Moreover, $\dot{\vec{q}} \cdot \vec{F} = 0$ at all times. In this case the force only changes the direction, and not the modulus of the velocity.

3rd Law

Newton's third law states that the reference frame does not matter for the description of the evolution of two particles, even when they interact with each other — i.e. when they exert forces on each other. Consider for instance the motion of two particles of the same mass m that reside at the positions $\vec{q}_1(t)$ and $\vec{q}_2(t)$. We decide to observe them from a position right in the middle between the two particles $\vec{Q} = \big(\vec{q}_1(t) + \vec{q}_2(t)\big)/2$. In accordance with observation, we require that this is an inertial frame of reference in the absence of external forces, such that $\ddot{\vec{Q}} = \vec{0}$ according to Newton's first law. However, Newton's second law implies that also

$$\vec{0} = 2m\ddot{Q} = m\,\ddot{\vec{q}}_1 + m\,\ddot{\vec{q}}_2 = \vec{F}_1 + \vec{F}_2$$

where $\vec{F}_1 = m\,\ddot{\vec{q}}_1$ and $\vec{F}_2 = m\,\ddot{\vec{q}}_2$ are the forces acting on particle 1 and 2, respectively. Up to a change of sign the forces are the same, $\vec{F}_1 = -\vec{F}_2$. This action-reaction principle is stipulated by

Axiom 3.3: Newton's 3rd law

Forces act in pairs:

actio when a body A is pushing a body B with force $\vec{F}_{A\to B}$

reactio then the body B is pushing A with force $\vec{F}_{B\to A} = -\vec{F}_{A\to B}$

The forces in such a pair are always balanced, $\vec{F}_{A\to B} + \vec{F}_{B\to A} = \vec{0}$.

Example 3.3: Fixing a hammok at a tree

When you lie in a hammok that is fixed at a tree, your hammok exerts a force $\vec{F}_h$ on the tree (actio). The hammok stays where it is because the tree pulls back with exactly the same force $-\vec{F}_h$, up to a change of sign (reactio).

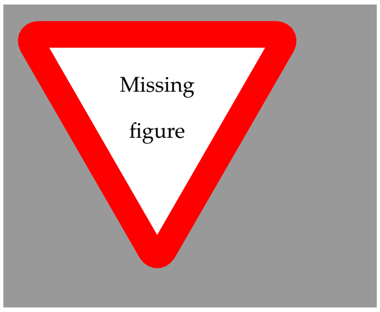


Figure 3.1: Graphical illustrations of forces involved in hanging a hammat on a tree.

Example 3.4: Ice scaters

- When two ice scaters of the same mass push each other starting from a position at rest, then they will move in opposite directions with the same speed (unless they break).
- When they have masses m_1 and m_2 their velocities will be related by $m_1\,\vec{v}_1 + m_2\,\vec{v}_2 = \vec{0}$ because $\vec{v}_1 = \vec{v}_2 = \vec{0}$ initially, and $m_1\,\dot{\vec{v}}_1 + m_2\,\dot{\vec{v}}_2 = \vec{F}_1 + \vec{F}_2 = \vec{0}$ at any instant of time.

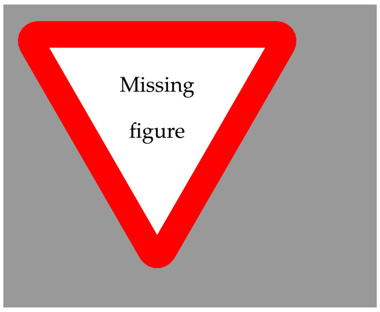


Figure 3.2: Graphical illustrations of motion of the two ice-scaters of Example 3.4.

Punchline

Newton's equations are stated nowadays in terms of derivatives, a concept in calculus that has been pioneered by Leibnitz.[2] In this language they take the following form for a particle of mass m that

[2] Even thought the principles have been understood by Newton which lead to a very long fight for authorship and fame.

is at position $\vec{q}(t)$ at time t,

$$\begin{aligned}\dot{\vec{q}}(t) &= \vec{v}(t) \\ \dot{\vec{v}}(t) &= \frac{1}{m}\,\vec{F}_{\text{tot}}(\vec{q}(t), \vec{v}(t), t)\end{aligned}$$

Prior to Newton physical theories adopted the aristotelian point of view that $\vec{v}$ is proportional to the force. Indceed in those days many scientists were regularly inspecting mines, and from the perspective of pushing minecarts is is quite natural to assert that their velocity is proportional to the pushing force. Galilei's achievement is to add the 'tot' of the force side of the equation, pointing out that there also is a friction force acting on the minecart. Newton's achievement is to add the 'dot' on the left side of the equation, stating that the velocity stays constant when the pushing force and the friction force balance.

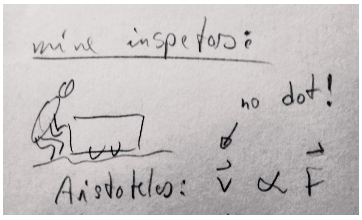


Figure 3.3: Illustration about the pre-newtonian understanding of the relation between force and the velocity of a body.

Example 3.5: Pushing a minecart

The motion of the minecart is one-dimensional along its track such that the position, q, velocity, x, and forces are one-dimensional, i.e., scalar functions. Once the minecart is moving it experiences a friction force $F_f = -\gamma\, v$, that is growing with speed, v. Now, let the mineworker push with a constant force F_M. Consequently,

$$\ddot{q} = \dot{v} = F_{\text{tot}} = F_M - \gamma\, v$$

The minecart travels with constant velocity $\dot{v} = 0$, when the attacking forces balance, i.e., for $v_c = F_M/\gamma$. However, when starting from a different velocity, $v(t_0) = v_0$, we rather find an exponential approach to the asymptotic velocity,

$$v(t) = v_c + \big(v_0 - v_c\big)\ \mathrm{e}^{-\gamma\,(t-t_0)}$$

After all, $v(t_0) = v_c + \big(v_0 - v_c\big) = v_0$ and

$$\dot{v}(t) = \big(v_0 - v_c\big)\ (-\gamma)\, \mathrm{e}^{-\gamma\,(t-t_0)} = -\gamma\ \big(v(t) - v_c\big) = -\gamma\ v(t) + F_M$$

The advantage of the Newtonian approach above earlier modeling attempts is that is makes a quantitative prediction about the asymptotic velocity, and that it also addresses the regime where the velocity is changing, e.g., when the minecart is taking up speed.

3.4 *Constants of motion (CM)*

In the previous section we saw that Newton's laws can be expressed as equations relating the second derivative of the position of a particle to the forces acting on the particle. The forces are determined as part of setting up the physical model. Subsequently, determining the time dependence of the position is a mathematical problem. Often it can be solved by finding constraints on the solution that must hold for all times. Such a constraint is called a

Definition 3.3: Constant of motion

A function $\mathcal{C}(\vec{q},\dot{\vec{q}},t)$ is a *constant of motion (CM)* iff its time derivative vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\vec{q},\dot{\vec{q}},t) = 0$$

It provides us with an opportunity to take a closer look at the expressions that emerge when taking derivatives of functions with arguments that are vectors. In order to evaluate the time derivative of $\mathcal{C}$ we write $\vec{q} = (q_1, \ldots\ldots, q_D)$, and apply the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\vec{q}(t),\dot{\vec{q}}(t),t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(q_1(t)\cdots q_D(t),\dot{q}_1(t)\cdots\dot{q}_D(t),t)\\ &= \sum_{i=1}^{D}\frac{\mathrm{d}q_i}{\mathrm{d}t}\,\frac{\partial\mathcal{C}}{\partial q_i}\bigg|_{\substack{q_j \text{ with } j\neq i\\ \dot{\vec{q}},\,\mathrm{t}}} + \sum_{i=1}^{D}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\,\frac{\partial\mathcal{C}}{\partial\dot{q}_i}\bigg|_{\substack{\vec{q},\,t\\ \dot{q}_j \text{ with } j\neq i}} + \frac{\partial\mathcal{C}}{\partial t}\bigg|_{\vec{q},\dot{\vec{q}}}\end{aligned}$$

In this expression the operator ∂ is called 'partial', and the derivative $\partial\mathcal{C}/\partial q_i\,\big|_{\text{conditions}}$ is denoted as 'partial derivative of $\mathcal{C}$ with respect to q_i for fixed ...' arguments that are explicitly indicated in the conditions. In other words, for the pupose of calculating the partial derivative, we consider $\mathcal{C}$ to be a function of only the single argument q_i based on the conditions spelled out as subscript after the vertical bar $\big|$. For sake of a more compact notation we also write $\partial_{q_i}\mathcal{C}$ rather than $\partial\mathcal{C}/\partial q_i$. Moreover, when it is clear from the context which conditions are adpopted, they will typically not be specified. From a physicists perspective this reduces clutter in the equations.

An more compact notation that is still more transparent is achieved by observing that the expressions in the sums amount to writing out in components a scalar product of $\vec{q}$ and $\dot{\vec{q}}$ with vectors that are obtained by the partial derivatives. These vectors are denoted *gradients* with respect to $\vec{q}$ and $\dot{\vec{q}}$, and they will be written as

$$\nabla_{\vec{q}} \mathcal{C} = \begin{pmatrix} \partial_{q_1} \mathcal{C} \\ \vdots \\ \partial_{q_D} \mathcal{C} \end{pmatrix} \qquad \text{and} \qquad \nabla_{\dot{\vec{q}}} \mathcal{C} = \begin{pmatrix} \partial_{\dot{q}_1} \mathcal{C} \\ \vdots \\ \partial_{\dot{q}_D} \mathcal{C} \end{pmatrix}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}(\vec{q}(t), \dot{\vec{q}}(t), t) = \dot{\vec{q}} \cdot \nabla_{\vec{q}} \mathcal{C} + \ddot{\vec{q}} \cdot \nabla_{\dot{\vec{q}}} \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

In terms of the phase-space coordinates $\Gamma = (\vec{q}, \dot{\vec{q}})$ one can even adopt the even more compact notation

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}(\vec{q}(t), \dot{\vec{q}}(t), t) = \dot{\vec{q}} \cdot \nabla_{\Gamma} \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

We conclude the section by introducing some important physical quantities that are constants of the motion in specific settings.

The kinetic energy

When no forces are acting on a particle, $\vec{F}_{\text{tot}} = \vec{0}$, it moves with constant velocity. All functions that depend only on the velocity will then be constant. In particular this holds for the kinetic energy, T, that will play a very important role in the following.

Theorem 3.2: Conservation of kinetic energy

The *kinetic energy* $T = \frac{m}{2} \dot{\vec{q}}^2$ of a particle is conserved iff no net force acts on the particle, i.e., iff $\vec{F}_{\text{tot}} = \vec{0}$.

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} T &= \frac{m}{2} \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \dot{q}_i \cdot \dot{q}_i \\ &= m \sum_i \dot{\vec{q}} \cdot \ddot{\vec{q}} = \dot{\vec{q}} \cdot (m \, \ddot{\vec{q}}) = \dot{\vec{q}} \cdot \vec{F}_{\text{tot}} = 0 \end{aligned}$$

In the last two steps we used Newton's 2nd law, and the assumption that $\vec{F}_{\text{tot}} = \vec{0}$. □

Work and total energy

From a physics perspective, work is performed when a body is moved that is experiencing an external force.

- When the force $\vec{F}$ is constant along a path of displacement $\vec{s} = \vec{q}_1 - \vec{q}_0$, from a position $\vec{q}_0$ to the position $\vec{q}_1$, then the work W amounts to the scalar product $W = \vec{F} \cdot \vec{s}$.

- When the force changes upon moving along the path, we parameterize the motion along the path by time, $\vec{q}(t)$, with $\vec{q}(t_0) = \vec{q}_0$ and $\vec{q}(t_1) = \vec{q}_1$ and break it into sufficiently small pieces $\vec{s}_i = \dot{\vec{q}}(t_i)\,\Delta t$ where the force $\vec{F}_i = \vec{F}(t_i)$ may be assumed to be constant. Then

$$W = \sum_i \vec{F}_i \cdot \vec{s}_i = \lim_{\Delta t \to 0} \vec{F}_i \cdot \dot{\vec{q}}\,\Delta t = \int_{t_0}^{t_1} \vec{F}(t) \cdot \dot{\vec{q}}(t)\,\mathrm{d}t = \int_{\vec{q}(t)} \vec{F} \cdot \mathrm{d}\vec{q}$$

  The last equality should be understood here as a definition of the final expression that is interpreted here in the spirit of the substitution rule of integration.

Definition 3.4: Work and Line Integrals

The *work*, W, of a particle that performs a path $\vec{q}$ under the influence of a force $\vec{F}(t)$ amounts to the result of the *line integral*

$$W = \int_{\vec{q}} \vec{F} \cdot \mathrm{d}\vec{q}$$

When the path is parameterized by time, then W amounts to the time integral of dissipated power $P(t) = \vec{F}(t) \cdot \dot{\vec{q}}(t)$,

$$W = \int \vec{F}(t) \cdot \dot{\vec{q}}(t)\,\mathrm{d}t = \int P(t)\,\mathrm{d}t$$

Remark 3.2. a) The scalar product $\vec{F} \cdot \mathrm{d}\vec{q}$ or $P(t) = \vec{F}(t) \cdot \dot{\vec{q}}(t)$ singles out only the action of the force parallel to the trajectory, when evaluating the work. The perperdicular component of the force changes the direction of motion, but it does not perform work.

b) A force that is always acting perpendicular to the velocity, i.e., perpendicular to the path of the particle, does not perform any work,

$$W = \int \vec{F}(t) \cdot \dot{\vec{q}}(t)\,\mathrm{d}t = \int 0\,\mathrm{d}t = 0$$

For some forces the work depends only on the initial and on the final point of the trajectory. They are called conservative forces because they can be used to define a total energy which is a constant of motion, i.e., it is conserved during the motion.

Definition 3.5: Conservative Force

A force $\vec{F}$ is called conservative if it can be written as minus the gradient of a potential, $\Phi(\vec{q})$

$$\vec{F}(\vec{q}) = -\nabla\Phi(\vec{q}) = -\begin{pmatrix} \partial_{q_1}\Phi \\ \vdots \\ \partial_{q_D}\Phi \end{pmatrix}$$

Remark 3.3. Conservative forces only depend on position, $\vec{F} = \vec{F}(\vec{q})$. They neither explicitly depend on time nor on the velocity $\dot{\vec{q}}$.

Theorem 3.3: Work for conservative forces

For conservative forces, $\vec{F} = -\nabla\Phi(\vec{q})$, the work for a path $\vec{q}(t)$ from $\vec{q}_0$ to $\vec{q}_1$ amounts to the difference of the potential evaluated at the initial and at the final point of the path

$$W = \int_{\vec{q}(t)} \vec{F}\cdot \mathrm{d}\vec{q} = \Phi(\vec{q}_0) - \Phi(\vec{q}_1)$$

Proof.

$$\begin{aligned} W &= \int_{t_0}^{t_1} \vec{F}\cdot\dot{\vec{q}}\,\mathrm{d}t = -\int_{t_0}^{t_1} \nabla\Phi\cdot\,\dot{\vec{q}}\,\mathrm{d}t = -\int_{t_0}^{t_1} \sum_i \frac{\partial\Phi}{\partial q_i}\,\frac{\partial q_i}{\partial t}\,\mathrm{d}t \\ &= -\int_{t_0}^{t_1} \frac{\mathrm{d}\,\Phi}{\mathrm{d}t}\,\mathrm{d}t = -\big(\Phi(\vec{q}(t_1)) - \Phi(\vec{q}(t_0))\big) = \Phi(\vec{q}_0) - \Phi(\vec{q}_1) \end{aligned}$$

□

Remark 3.4. The work performed along a closed path vanishes for conservative forces. After all, in that case $\vec{q}_1 = \vec{q}_0$ such that $W = \Phi(\vec{q}_0) - \Phi(\vec{q}_1) = 0$.

Theorem 3.4: Conservation of the total energy

The *total energy* $E = T + \Phi$ of a particle is conserved if it moves in a conservative force field $\vec{F} = -\nabla\Phi$.

Proof.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}\Phi}{\mathrm{d}t} = m\,\dot{\vec{q}}\cdot\ddot{\vec{q}} + \nabla\Phi\cdot\dot{\vec{q}} = \dot{\vec{q}}\cdot\underbrace{\big(m\ddot{\vec{q}} - \vec{F}\big)}_{=\,\vec{0}} = \vec{0}$$

In the third equality we used that the force is conservative, and in the final step, we used Newton's second law which states that $m\ddot{\vec{q}} = \vec{F}$. □

Momentum

Theorem 3.5: Conservation of momentum

The *momentum* $\vec{P} = \sum_{i=1}^{N} m_i \dot{\vec{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\vec{q}_i(t)$ is conserved if no net force $\vec{F}_{\text{tot}}$ acts on the system.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P} = \sum_{i=1}^{N} m_i \ddot{\vec{q}}_i(t) = \sum_{i<j} (\vec{f}_{ij} + \vec{f}_{ji}) + \sum_i \vec{F}_i = \vec{F}_{\text{tot}} = \vec{0}$$

where $\vec{f}_{ij} + \vec{f}_{ji}$ vanishes due to Newton's third law, and the net extanal force is zero by assymption. □

add example: particle collisions in 1d, different mass

Angular Momentum

Theorem 3.6: Conservation of momentum

The *angular momentum* $\vec{L} = \sum_{i=1}^{N} m_i \vec{q}_i(t) \times \dot{\vec{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\vec{q}_i(t)$ is conserved if no external forces act on the system and if the interaction forces between pairs of particles act parallel to the line connecting the particles.

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{L} &= \sum_{i=1}^{N} m_i \Big(\dot{\vec{q}}_i(t) \times \dot{\vec{q}}_i(t) + \vec{q}_i(t) \times \ddot{\vec{q}}_i(t) \Big) \\ &= \sum_{i<j} \Big(\vec{q}_i(t) \times \vec{f}_{ij} + \vec{q}_j(t) \times \vec{f}_{ji} \Big) = \sum_{i<j} \Big(\vec{q}_i(t) - \vec{q}_j(t) \Big) \times \vec{f}_{ij} = \vec{0}\end{aligned}$$

where we used that $\vec{f}_{ij} = -\vec{f}_{ji}$ due to Newton's third law, and that $(\vec{q}_i(t) - \vec{q}_j(t))$ is parallel to $\vec{f}_{ij}$ by assumption on the particle interactions. □

add example: particle collisions in 3d

3.5 *Falling men and cat — a worked example*

When a cat, that has a mass of $m = 3\,\mathrm{kg}$, falls from a balcony in the fourth floor, i.e., from a height $H \simeq 4 \times 3\,\mathrm{m} = 12\,\mathrm{m}$, the initial potential energy

$$V_{\text{cat}} = mgH = 3\,\text{kg} \times 10\,\text{m/s}^2 \times 12\,\text{m} = 360\,\text{kg}\,\text{m}^2/\text{s}^2$$

will be transformed into kinetic energy and then dissipated when the cat hits the ground.

To get an idea about this energy we compare it to the energy dissipated when a man of mass $M = 80\,\text{kg}$, falls out of his bed that has a height of $h = 50\,\text{cm}$,

$$V_{\text{man}} = Mgh = 80\,\text{kg} \times 10\,\text{m/s}^2 \times 0.5\,\text{m} = 400\,\text{kg}\,\text{m}^2/\text{s}^2$$

Apparently, from the point of the dissipated energy the fall of the cat is not as bad as it looks from first sight.

3.6 *Accidents on work and street — a worked example*

A paramedic emergency ambulence receives two calls from an accident site:

a craftsmen has fallen from a roof with a height H

a teenager has hits a tree with his motorcycle with a speed v

Discuss the height of the fall where the energy of the craftsman approximately matches those of the motor cyclist when the latter is driving in the city, $v_C = 50\,\text{km/h}$, outside the city, $v_L = 100\,\text{km/h}$, on a German autobahn with $v_A = 150\,\text{km/h}$ or really speeding with $v_S = 200\,\text{km/h}$. We assume that they both have comparable mass.

In the case of the craftsman the energy dissipated, when he hits the ground will be

$$mgH = V_{\text{worker}} = T_{\text{motorcyclist}} = \frac{m}{2}\,v^2 \quad \Leftrightarrow \quad H = \frac{v^2}{2g}$$

Hence we find

v	50 km/h	100 km/h	150 km/h	200 km/h
H	12 m	50 m	110 m	200 m
floor	4	16	36	64

The motorcyclist will most likely encounter seriously more sever injuries, unless the craftsman is working on a really high building.

3.7 Problems

Rehearsing Concepts

Problem 3.1. **Derivatives of Elementary Functions**

Determine the derivatives of the following functions.

a) $\sin x$

b) $\cos x$

c) $\tan x$

d) x^n

e) $\sinh x$

f) $\cosh x$

g) $\tanh x$

h) e^x

i) $\ln x$

$\log_{10} x$

n^x

x^x

Problem 3.2. **Integrals of Elementary Functions**

Evaluate the following integrals.

a) $\displaystyle\int_{-1}^{1} \mathrm{d}x\,(a+x)^2$

b) $\displaystyle\int_{-5}^{5} \mathrm{d}q\,(a+b\,q^3)$

$\displaystyle\int_{0}^{B} \mathrm{d}k\,\tanh^2(kx)$

c) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\,\mathrm{e}^{-x/L}$

d) $\displaystyle\int_{-L}^{L} \mathrm{d}y\,\mathrm{e}^{-y/\xi}$

e) $\displaystyle\int_{0}^{L} \mathrm{d}z\,\frac{z}{a+b\,z^2}$

f) $\displaystyle\int_{0}^{\infty} \mathrm{d}x\,x\,\mathrm{e}^{-x^2/(2Dt)}$

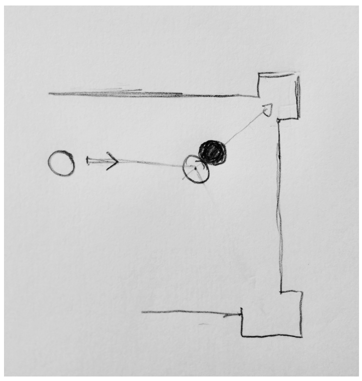
g) $\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}\ell\,\ell\,\mathrm{e}^{-\ell^2/(2Dt)}$

$\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}} \mathrm{d}z\,x\,\mathrm{e}^{-z\,x^2}$

Except for the integration variable all quantities are considered to be constant.

Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

Problem 3.3. **Collisions on a billiard table.**



The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity $\vec{v}$), and hit the black ball such that it ends up in pocket to the top right.

What is tricky about the sketched track?

What might be a better alternative?

Problem 3.4. **Pulling a Duck.**

A child is pulling a toy duck with a force of $F = 5\,\mathrm{N}$. The duck has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.[3]

a) Describe the motion of the duck when there is no friction.
In the beginning the duck is at rest.

[3] For this angle one has $\tan\theta \approx 3/4$.

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the duck.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

Practicing Concepts

Problem 3.5. **Derivatives of Common Composite Expressions**

Evaluate the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}(a+x)^b$

b) $\frac{\partial}{\partial x}(x+b\,y)^2$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x+y(x))^2$

d) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta(t)$

e) $\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin\theta(t)\ \cos\theta(t)\big)$

f) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\big(2\theta(t)\big)$

g) $\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{a+b\,z^2}$

h) $\frac{\partial}{\partial x_3}\left[\sum_{j=1}^{6} x_j^2\right]^{-1/2}$

i) $\frac{\partial}{\partial y_1}\ln(\vec{x}\cdot\vec{y})$

In these expressions a and b are real constants, and $\vec{x}$ and $\vec{y}$ are 6-dimensional vectors.

Problem 3.6. **Solving Integrals by Partial Integration**

Evaluate the following integrals by partial integration

$$\int \mathrm{d}x\, f(x)\, g'(x) = f(x)\, g(x) - \int \mathrm{d}x\, f'(x)\, g(x)$$

a) $\int_a^b \mathrm{d}x\, x\, \mathrm{e}^{kx}$

b) $\int_a^b \mathrm{d}x\, x^2\, \mathrm{e}^{kx}$

c) $\int_a^b \mathrm{d}x\, x^n\, \mathrm{e}^{kx}$, $n \in \mathbb{N}$

The integral c) can only be given as a sum over $j = 0, \dots, n$.

Problem 3.7. **Substitution with Trigonometric and Hyperbolic Functions**

Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{q(x_1)}^{q(x_2)} \mathrm{d}q\, f(q) = \int_{x_1}^{x_2} \mathrm{d}x\, q'(x)\, f(q(x))$$

with a function $q(x)$ that is bijective on the integration interval $[x_1, x_2]$.

a) $\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}}$ by substituting $x = \sin\theta$

b) $\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}}$ by substituting $x = \sinh z$

c) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1+x^2}$ by substituting $x = \tan\theta$

d) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1-x^2}$ by substituting $x = \tanh z$

Problem 3.8. **Crossing a River.**

A ferry is towed at the bank of a river of width $B = 100$ m that is flowing at a velocity $v_F = 4$ m/s to the right. At time $t = 0$ s it departs and is heading with a constant velocity $v_B = 10$ km/h to the opposite bank.

a) When will it arrive at the other bank when it always heads straight to the other side? (In other words, at any time its velocity is perpendicular to the river bank.)

   How far will it drift downstream on its journey?

b) In which direction (i.e. angle of velocity relative to the downstream velocity of the river) must the ferryman head to reach exactly at the opposite side of the river?

   Determine first the general solution. What happens when you try to evaluate it for the given velocities?

Problem 3.9. **Retroreflector paths on bike wheels.**

The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retroreflectors fixed in the sparks enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio ρ of the reflectors distance d to the wheel axis and the wheel radius r. A small stone in the profile traces a cycloid ($\rho = 1$). Animations of the trajectories can be found at
https://en.wikipedia.org/wiki/Trochoid and http://katgym.by.lo-net2.de/c.wolfseher/web/zykloiden/zykloiden.html.

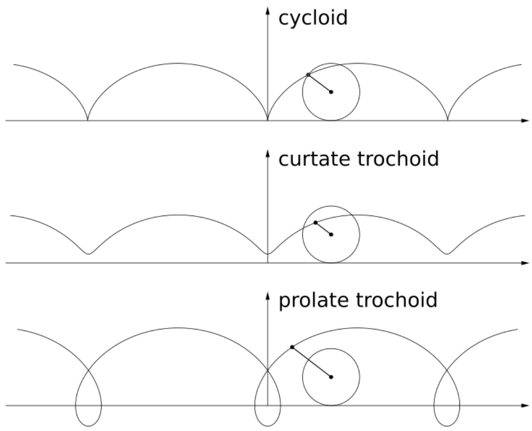


Figure 3.4: [Wikimedia CC BY 4.0 (modified)]

A trochoid is most easily described in two steps: Let $\vec{M}(\theta)$ be the position of the center of the disk, and $\vec{D}(\theta)$ the vector from the center to the position $\vec{q}(\theta)$ that we follow (i.e. the position of the retroreflector) such that $\vec{q}(\theta) = \vec{M}(\theta) + \vec{D}(\theta)$.

a) The point of contact of the wheel with the street at the initial time t_0 is the origin of the coordinate system. Moreover, we single out one spark and denote the change of its angle with respect to its initial position as θ. Note that negative angles θ describe forward motion of the wheel!

Sketch the setup and show that

$$\vec{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix} , \qquad \vec{D}(\theta) = \begin{pmatrix} -d\, \sin(\varphi+\theta) \\ d\, \cos(\varphi+\theta) \end{pmatrix} .$$

check signs of components of $\vec{D}$

What is the meaning of φ in this equation?

b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time, $L = \int_{t_0}^{t} \mathrm{d}t \; \left|\dot{\vec{q}}(\theta(t))\right|$. Show that therefore

$$L = r \int_0^\theta \mathrm{d}\theta \, \sqrt{1+\rho^2+2\rho\, \cos(\varphi+\theta)}$$

c) Consider now the case of a cycloid and use $\cos(2x) = \cos^2 x - \sin^2 x$ to show that the expression for L can then be written as

$$L = 2\,r \int_0^\theta \mathrm{d}\theta \, \left| \cos \frac{\varphi+\theta}{2} \right|$$

How long is one period of the track traced out by a stone picked up by the wheel profile?

Proofs

Transfer and Bonus Problems, Riddles

Problem 3.10. **Running Mothers.**

In the Clara Zetkin Park one regularly encounters blessings[4] of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their youngs in pouches.

[4] Look up "terms of venery" if you ever run out of collective nouns.

a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.
The carriages suffer from friction. Let the friction coefficient be $\gamma = 0.3$.
When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.

b) How does the running speed matter in this discussion?

c) How does the mass of the babies/youngs make a difference?

Problem 3.11. **Hypotrochoids, roulettes, and the spirograph.**

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

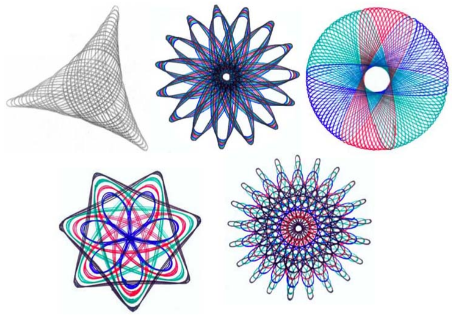


Figure 3.5: Wikimedia Public domain]

a) Consider the track of a pole attached to a disk with n cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the curves developed in Problem 3.9 such that you can describe hypotrochoids.

c) Test your result by writing a Python program that plots the cruves for given m and n.

4
Equations of Motion

In Chapter 3 we learned how to set up a physical model based on finding the forces acting on a body, and thus determining the acceleration of its motion. For a particle of mass m and position $\vec{q}$ resulting equation relates its acceleration $\ddot{\vec{q}}$ to the force, that in itself may depend on $\vec{q}$, $\dot{\vec{q}}$, and t. The resulting equation

$$m\,\ddot{\vec{q}} = \vec{F}(\vec{q}, \dot{\vec{q}}, t) \tag{4.0.1}$$

is referred to as the *equation of motion* (EOM) of the particle. From the mathematical point of view it is an *ordinary differential equation* (ODE). The present chapter will

i. introduce ODEs,

ii. discuss strategies to characterize sets of solutions and find solutions for specific initial conditions, and

iii. discuss the solutions of examples that are of particular relevance in physics.

The methods will be introduced and motivated based on elementary physical problems.

At the end of this chapter we will be able to discuss the motion of plants around the sun and moons arount their planets — as well as the fate of a man that lost his electromagnetic interaction with Earth.

add phase-space portraits: explanation and for all examples

4.1 Motivation and Outline: What are EOMs and ODEs?

The *order* of a differential equation denotes the highest order derivative that appears in the equation. It is called an *ordinary dif-*

ferential equation, when all derivatives are taken with respect to the same variable. From this perspective Equation (4.0.1) is a ordinary differential equation (ODE) of second order. It is called *autonomous* when its right-hand side does not explicitly depend on time.

For N particles with masses m_i, $i = 1 \cdots N$ that are moving in D dimensions we will have a system of N differential equations for the D dimensional vectors $\vec{q}_i = (q_{i,\alpha}, \alpha = 1 \cdots D)$

$$\ddot{q}_{i,\alpha} = \frac{1}{m_i} F_{i,\alpha}(\{\vec{q}_i, \dot{\vec{q}}_i\}_{i=1\cdots N}, t)\,, \qquad i = 1 \cdots N\,, \quad \alpha = 1 \cdots D$$

By introducing the variables $\vec{v}_i = \dot{\vec{q}}_i$ the EOMs can always be written as a a set of $2N$ first order ODE

$$\begin{aligned} \dot{q}_{i,\alpha} &= v_{i,\alpha} \\ \dot{v}_{i,\alpha} &= \frac{1}{m_i} F_{i,\alpha}(\{\vec{q}_i, \dot{\vec{q}}_i\}_{i=1\cdots N}, t) \end{aligned}$$

For an autonomous system this can be written in a more compact form by introducing the $2DN$ dimensional phase-space coordinate $\vec{\Gamma}$ and the flow $\vec{\mathcal{V}}$ as follows

$$\begin{aligned} \vec{\Gamma} &= \left(q_{1,1} \cdots q_{1,D}\,,\ q_{2,1} \cdots q_{N,D}\,,\ \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\ \dot{q}_{2,1} \cdots \dot{q}_{N,D}\right) \\ \vec{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\ v_{2,1} \cdots v_{N,D}\,,\ \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\ \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\right) \\ \dot{\vec{\Gamma}} &= \vec{\mathcal{V}}(\vec{\Gamma}) \qquad \text{for autonomous systems.} \end{aligned}$$

Moreover, a non-autonomous system can always be expressed as an autonomous, first order ODE where $\vec{\gamma}$ and $\vec{\mathcal{V}}$ denote points in a $2DN + 1$ dimensional phase space, however,

$$\begin{aligned} \vec{\Gamma} &= \left(q_{1,1} \cdots q_{1,D}\,,\ q_{2,1} \cdots q_{N,D}\,,\ \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\ \dot{q}_{2,1} \cdots \dot{q}_{N,D}\,,\ t\right) \\ \vec{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\ v_{2,1} \cdots v_{N,D}\,,\ \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\ \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\,,\ 1\right) \\ \dot{\vec{\Gamma}} &= \vec{\mathcal{V}}(\vec{\Gamma}) \qquad \text{for non-autonomous systems.} \end{aligned}$$

In phase space, Γ denotes a point that characterizes the state of our system, and $\vec{\mathcal{V}}(\Gamma)$ provides the *unique* direction and velocity of the temporal change of this state. In a crude approximation, that is accurate however for sufficiently small Δt, we have

$$\vec{\Gamma}(t + \Delta t) \simeq \vec{\Gamma}(t) + \Delta t\ \vec{\mathcal{V}}(\vec{\Gamma}(t))$$

This admits a graphical representation of the solutions of the ODE that we will explore in the next section when we discuss the solutions of some ODEs that commonly arise in mechanical problems.

4.2 *Free flight*

We first discussion the motion of a single particle that is moving in a gravitational field giving rise to the constant gravitational acceleration $\vec{g}$. Hence, the particle position $\vec{q}(t)$ obeys the EOM

$$\ddot{\vec{q}} = \vec{g} \tag{4.2.1}$$

The right hand side of this equation does not depend on $\vec{q}$. This has two remarkable consequences that we will exploit whenever possible.

Decoupling of the motion of different DOF

Each component q_α of $\vec{q}$ can be solved independently of the other DOF

$$\dot{q}_\alpha = g_\alpha$$

Rather that dealing with a vector-valued ODE, one can therefore solve D scalar ODEs which turns out to be a much simpler task. Indeed, we will see in our further discussion that he solution of vector-valued ODEs will often proceed via a coordinate transformation that decouples the different DOF.

The ODE can be integrated

The ODE, Equation (4.2.1), can be solved by integration

Algorithm 4.1: Integrating ODEs

An ODE for $\vec{f}(t)$ can be solved by *integration* when its right-hand side does not depend on on $\vec{f}(t)$, i.e., when it takes the form

$$\dot{\vec{f}}(t) = \vec{v}(t)$$

For the initial condition $\vec{f}(t_0) = \vec{f}_0$ one obtains then

$$\vec{f}(t) = \vec{f}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \dot{\vec{f}}(t') = \vec{f}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \vec{v}(t')$$

which expresses the solution of the ODE in terms of an integral.

In particular, when the ODE is autonomous we obtain

$$\vec{f}(t) = \vec{f}_0 + \vec{v}\,(t - t_0)$$

For the free flight we thus obtain for an initial conditions $\vec{q}(t_0) = \vec{q}_0$ and $\dot{\vec{q}}(t_0) = \vec{v}_0$, that

$$\dot{\vec{q}}(t) = \vec{v}_0 + \int_{t_0}^{t} \mathrm{d}t' \vec{g} = \vec{v}_0 + \vec{g}\,(t - t_0)$$

and

$$\begin{aligned}
\vec{q}(t) &= \vec{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \dot{\vec{q}}(t) = \vec{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \left(\vec{v}_0 + \vec{g}\,(t - t_0)\right) \\
&= \vec{q}_0 + \vec{v}_0 \int_{t_0}^{t} \mathrm{d}t' + \vec{g} \int_{t_0}^{t} \mathrm{d}t'\,(t - t_0) \\
&= \vec{q}_0 + \vec{v}_0\,(t - t_0) + \vec{g} \int_{0}^{t-t_0} \mathrm{d}t''\,(t - t_0) \\
&= \vec{q}_0 + \vec{v}_0\,(t - t_0) + \frac{1}{2}\,\vec{g}\,(t - t_0)^2
\end{aligned}$$

When we denote the direction anti-parallel to $\vec{g}$ as $z = q_1$, then

$$\begin{aligned}
z(t) = q_1(t) &= z(t_0) + v_z(t_0)\,(t - t_0) - \frac{g}{2}\,(t - t_0)^2 \\
q_i(t) &= q_i(t_0) + v_i(t_0)\,(t - t_0)\,, \quad \text{for } i > 1
\end{aligned}$$

In phase space these equations are represented by the flow Non-dimensionalization leads in this case to another interesting finding. Let us measure positions $q_i(t)$ in multiples of $q_i(t_0)$, velocities in multiples of $v_i(t_0)$, and introduce the dimensionless time $\tau =$

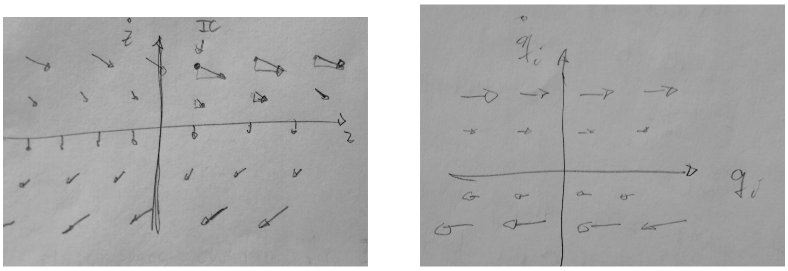


Figure 4.1: Phase-space flow for motion with constant acceleration (left) and for free flight (right).

$v_i(t_0)(t - t_0)/q_i(t_0)$. Then the solution of the EOM take the form

$$\hat{z}(\tau) = \frac{q_1(t)}{q_1(t_0)} = 1 + \tau - I\,\tau^2\,, \quad \text{with } I = \frac{g\,q_1(t_0)}{2\,v_i^2(t_0)}$$

$$\hat{q}_i(t) = \frac{q_i(t_0)}{q_i(t_0)} = 1 + \tau\,, \quad \text{for } i > 1$$

The second relation implies that in phase space $\hat{q}_i$ is a straight horizontal line through $\hat{v}_i = 1$, and from the first equation we find $tau = (1 - v)/2I$ such that all trajectories in phase space are parabola of the form

$$\hat{v}_z^2 = 1 - 4I\,(\hat{z} - 1) \tag{4.2.2}$$

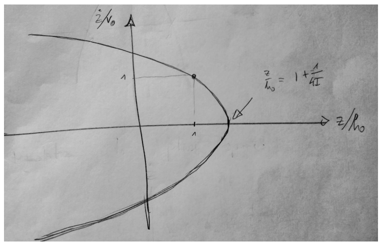


Figure 4.2: Sketch of the universal form of the free-flight trajectories in phase space, Equation (4.2.2).

4.3 *Free flight with Stokes friction*

The falling of a ball in a viscous medium can be described by the equations of motion

$$m\,\ddot{h}(t) = -m\,g - \mu\,\dot{h}(t)\,. \tag{4.3.1}$$

Here $h(t)$ is the vertical position of the ball (height), g is the acceleration due to gravity, and the contribution $-\mu\,\dot{h}(t)$ describes Stokes friction, i.e. the viscous drag on the ball. The viscosity $[\eta]$ of a fluid is measured in terms of Pa $=$ kg/ms. For air and water it takes values of about $\eta_{\text{air}} \simeq 2 \times 10^{-5}\,\text{kg/m s}$, and $\eta_{\text{water}} \simeq 1 \times 10^{-3}\,\text{kg/m s}$, respectively. The Stokes friction force only depends on the shape and velocity of the body. Hence, dimensional analysis implies that

$$\mu \propto R\,\eta$$

check factor

where R characterizes the size of the falling object. For a sphere of radius R the proportionality constant takes the value of $2/3$.

The Equation (4.3.1) can not be integrated by integration, Algorithm 4.1, because its right-hand side explicitly depends on $\dot{h}$. This equation of motion is best solved by separation of variables.

Algorithm 4.2: Separation of variables

A one-dimensional ODE for $f(t)$ can be solved by *separation* of variables when its right-hand side can be written as the product of factors that only involve $f(t)$ and another function of t, respectively, i.e., when it takes the form

$$\dot{f}(t) = g(f(t))\, h(t)$$

For the initial condition $\vec{f}(t_0) = \vec{f}_0$ one obtains then

$$\int_{t_0}^{t} \mathrm{d}t'\, h(t') = \int_{t_0}^{t} \mathrm{d}t'\, \frac{\dot{f}(t')}{g(f(t'))} = \int_{f_0}^{f(t)} \mathrm{d}f\, \frac{1}{g(f)}$$

which provides the solution in terms of two integrals.

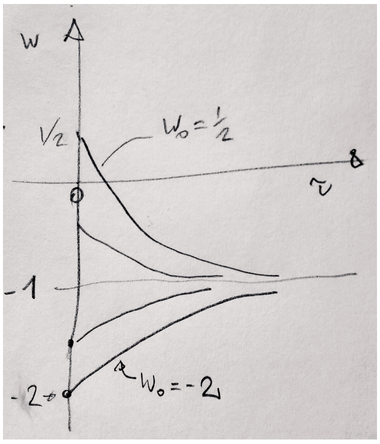


Figure 4.3: Sketch of $w(\tau)$ as obtained in Equation (4.3.2).

For Equation (4.3.1) we thus obtain the velocity $v(t) = \dot{h}(t)$ for an initial velocity v_0. In order to simplify notations we discuss its solution based on the dimensionless velocity, $w = \mu\dot{h}/m\,g$, and absorb the factor m/μ into the dimensionless time $\tau = \mu(t - t_0)/m$,

$$\frac{\mathrm{d}w(t)}{\mathrm{d}\tau} = \frac{m}{\mu}\frac{\mathrm{d}w(t)}{\mathrm{d}t} = -1 - w(t)$$

Separation of variables provides

$$\tau = \int_0^{\tau} \mathrm{d}\tau' = -\int_{w_0}^{w(\tau)} \mathrm{d}w\, \frac{1}{1+w} = -\ln\frac{1+w(\tau)}{1+w_0}$$

$$\Leftrightarrow \quad w(\tau) = -1 + \left(w_0 + 1\right) \mathrm{e}^{-\tau} \tag{4.3.2}$$

Stokes friction induces that for large times, $\tau 1$, the ball is sinking with the constant Stokes velocity. It takes the value -1 in our dimensionless units, and hence $v_\infty = -m\,g/\mu$ in terms of the physical units.

The position of the sphere can be obtained by integrating Equation (4.3.2) for $w(\tau) = \mathrm{d}\hat{h}/\mathrm{d}\tau$ with initial condition $\hat{h}_0$,

$$\hat{h}(\tau) = \hat{h}_0 + \int_0^{\tau} \mathrm{d}\tau\, \frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \hat{h}_0 + \int_0^{\tau} \mathrm{d}\tau\, \left(-1 + \left(w_0 + 1\right) \mathrm{e}^{-\tau}\right) \tag{4.3.3}$$

$$= \hat{h}_0 - \tau + \left(w_0 + 1\right)\ \left(1 - \mathrm{e}^{-\tau}\right) \tag{4.3.4}$$

or in terms of physical units

$$h(t) = h_0 - v_\infty\,(t - t_0) + \frac{m}{\mu}\,(v_0 - v_\infty)\,\left[1 - \exp\left(-\frac{\mu}{m}(t - t_0)\right)\right] \tag{4.3.5}$$

It is instructive to explore how the evolution with Stokes friction is related to the free flight $h_f(t) = h_0 + v_0(t - t_0) - g\,(t - t_0)^2$ obtained in Section 4.2. This can most effectively be done by Taylor expansion of Equation (4.3.3) for small τ and subsequently expressing the result in physical units. Based on the Taylor expansion of the exponential function $\mathrm{e}^{-\tau} = \sum_{n=0}^{\infty}(-\tau)^n/n!$ we thus find

explain Taylor expansion

$$\begin{aligned}
\hat{h}(\tau) &= \hat{h}_0 - \tau + (w_0 + 1)\,\left(\tau - \frac{\tau^2}{2} + \frac{\tau^3}{6} - \cdots\right) \\
&= \hat{h}_0 + w_0\,\tau - (w_0 + 1)\,\frac{\tau^2}{2}\,\left(1 - \frac{\tau}{3} + \cdots\right) \\
\Leftrightarrow \quad h(t) &= h_0 + v_0\,(t - t_0) - \frac{\mu}{m}\,\left(v_0 + \frac{m\,g}{\mu}\right)\,\frac{(t - t_0)^2}{2}\,\left(1 - \frac{\mu\,(t - t_0)}{3\,m} + \cdots\right) \\
&= h_0 + v_0\,(t - t_0) - \frac{g}{2}\,(t - t_0)^2\,\left(1 - \frac{v_0}{v_\infty}\right)\,\left(1 - \frac{\mu\,(t - t_0)}{3\,m} + \cdots\right)
\end{aligned}$$

This implies that Stokes friction provides a small corrections to the free flight if the initial velocity is small as compared to the asymptotic velocity of free flight, $|v_0| \ll v_\infty = m\,g/\mu$ provided that one restricts the attention to times that are small as compared to the time scale m/μ where the asymptotic velocity is reached. Equation (4.3.2) implies that this amounts to situations where the velocity $|v(t)|$ is small as compared to the Stokes settling speed v_∞.

Example 4.1: Stokes friction for a steel ball

A steel ball with a diameter of 1 cm has a mass of about

$$m = \frac{4\pi}{3}\, 2\times 10^{3}\,\mathrm{kg/m^3}\, \frac{1\times 10^{-6}\,\mathrm{m}^3}{8} \simeq 1\times 10^{-3}\,\mathrm{kg}$$

In air it will reach a terminal velocity of about

$$\begin{aligned} v_{\mathrm{air}} &= \frac{m\,g}{\mu_{\mathrm{air}}} = \frac{3\,m\,g}{2\,\eta_{\mathrm{air}}\,R} = \frac{3\times 1\times 10^{-3}\,\mathrm{kg}\,10\,\mathrm{m/s^2}}{2\times 2\times 10^{-5}\,\mathrm{kg/m\,s}\;1\times 10^{-2}\,\mathrm{m}} \\ &\simeq 7.5\times 10^{4}\,\mathrm{m/s} \end{aligned}$$

Saturation to this velocity occurs on time scales

$$t_{\mathrm{air}} = \frac{m}{\mu_{\mathrm{air}}} = \frac{m}{\eta_{\mathrm{air}}\,R} = \frac{1\times 10^{-3}\,\mathrm{kg}}{2\times 10^{-5}\,\mathrm{kg/m\,s}\;1\times 10^{-2}\,\mathrm{m}} = 5\times 10^{3}\,\mathrm{s}$$

and this time the bullet will have dropped by a distance $g\,t_c^2/2 = 2.5\times 10^{7}\,\mathrm{m}$ which is much more than the thickness of the atmosphere. We conclude that Stokes friction is not relevant for the motion of a bullet in air.
Even in water, where the viscosity is larger by a factor of 50, we will have

$$\begin{aligned} v_{\mathrm{water}} &= \frac{3\,m\,g}{2\,\eta_{\mathrm{water}}\,R} = \frac{3\times 1\times 10^{-3}\,\mathrm{kg}\,10\,\mathrm{m/s^2}}{2\times 1\times 10^{-3}\,\mathrm{kg/m\,s}\;1\times 10^{-2}\,\mathrm{m}} \\ &\simeq 1.5\times 10^{3}\,\mathrm{m/s} \end{aligned}$$

Saturation to this velocity occurs on time scales

$$t_{\mathrm{water}} = \frac{m}{\eta_{\mathrm{water}}\,R} = \frac{1\times 10^{-3}\,\mathrm{kg}}{1\times 10^{-3}\,\mathrm{kg/m\,s}\;1\times 10^{-2}\,\mathrm{m}} = 100\,\mathrm{s}$$

and this time the bullet will have dropped by a distance $g\,t_{\mathrm{water}}^2/2 = 5\times 10^{4}\,\mathrm{m}$ which is deeper than the deepest point in our Oceans.

Example 4.2: Stokes friction for sperms

Sperms are cells equipped with cilia that allow them to swim towards the egg for fertilization. They have a characteristic size L of a few micrometers and they swim in an environment that is approximated here as water. Their mass is of the order of $m_{\text{sperms}} = \rho_{\text{water}} L^3$. In this case their asymptotic speed is reached at a time scale

$$\begin{aligned} t_{\text{spermium}} &= \frac{m_{\text{spermium}}}{\mu_{\text{spermium}}} = \frac{\rho_{\text{water}} L^2}{\eta_{\text{water}}} \\ &= \frac{1 \times 10^3\,\text{kg/m}^3\; 1 \times 10^{-12}\,\text{m}^2}{1 \times 10^{-3}\,\text{kg/m\,s}} = 1 \times 10^{-6}\,\text{s} \end{aligned}$$

Stokes friction plays a major role for their swimming. See **?** for more details.

insert reference

4.4 *Free flight with turbulent friction*

In Example 4.1 we reached the puzzling conclusion that — for all physically relevant parameters — Stokes friction plays no role for the motion of a steel ball in air and water. On the other hand, we know from experience that friction arises to the very least for large velocities, like for gun shots. Friction arises because for large velocities the motion of the fluid around the ball goes turbulent, and the friction crosses over to a drag force with modulus

$$F_D = m \frac{\rho_{\text{fluid}}\, C_D}{8\, \rho_{\text{ball}}\, R}\, v^2 = m\, \kappa\, v^2$$

where m is the mass of the ball and C_D is a drag coefficient that typically takes values between 0.5 and 1. A very beautiful description of the physics of this equations has been provided in an instruction video by the NASA (click here to check it out).

To address motion affected by turbulent drag we measure time in units of $(\kappa g)^{-1/2}$ and velocity in units of g/κ. The dimensionless velocity $w(\tau)$ will then obey the equation of motion

$$\frac{\mathrm{d}w(\tau)}{\mathrm{d}\tau} = -1 - w^2(\tau)\, \text{sign}\big(w(\tau)\big)\,.$$

Which can again be solved by separation of variables

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{w_0}^{w(\tau)} \mathrm{d}w\, \frac{1}{1 + w^2\, \text{sign}\big(w(\tau)\big)}$$

First we consider an initial condition where $w_0 > 0$. We expect in that case that $w(\tau) > 0$ till some time τ_c, and then the particle will start falling due to the action of gravity. For $\tau < \tau_c$ we find

$$\begin{aligned} \tau &= -\int_{w_0}^{w(\tau)} \mathrm{d}w \, \frac{1}{1+w^2} = -\arctan(w(\tau)) - \arctan(w_0) \\ \Leftrightarrow \quad w(\tau) &= \tan\bigl(\arctan(w_0) - \tau\bigr) \end{aligned}$$

such that $\tau_c = \arctan(w_0)$. Moreover, for $\tau > \tau_c$ and $-1 < w(\tau) \leq 0$ we find

$$\begin{aligned} \tau - \tau_c &= -\int_{0}^{w(\tau)} \mathrm{d}w \, \frac{1}{1-w^2} = -\mathrm{atanh}(w_0) \\ \Rightarrow \quad w(\tau) &= \begin{cases} \tan\bigl(\arctan(w_0) - \tau\bigr) & \text{for} \quad \tau < \tau_c = \arctan(w_0) \\ \tanh\bigl(\arctan(w_0) - \tau\bigr) & \text{for} \quad \tau \geq \tau_c = \arctan(w_0) \end{cases} \end{aligned}$$

Similarly, when $W_0 < 0$ one obtains

$$w(\tau) = \begin{cases} \tanh\bigl(\mathrm{atanh}(w_0) - \tau\bigr) & \text{for} \quad 0 \geq w_0 > -1 \\ -1 & \text{for} \quad -1 = w_0 \\ \mathrm{cotanh}\bigl(\mathrm{acotanh}(w_0) - \tau\bigr) & \text{for} \quad -1 > w_0 \end{cases}$$

Range of applicability

Turbulent friction applies whenever

$$\mu\,|v| \lesssim m\,\kappa\,v^2 \quad \Leftrightarrow \quad |v| \gtrsim v_c = \frac{\mu}{m\kappa} \simeq \frac{\eta_{\text{fluid}}}{\rho_{\text{fluid}}\,R}$$

For the 1 cm steel ball considered in Example 4.1 the cross-over velocity v_c yields

$$v_c = \begin{cases} \frac{2\times 10^{-5}\,\mathrm{kg/m\,s}}{1\,\mathrm{kg/m^3}\times 1\times 10^{-2}\,\mathrm{m}} = 2\,\mathrm{mm/s} & \text{for air} \\ \frac{1\times 10^{-3}\,\mathrm{kg/m\,s}}{1\times 10^{3}\,\mathrm{kg/m^3}\times 1\times 10^{-2}\,\mathrm{m}} = 0.1\,\mathrm{mm/s} & \text{for water} \end{cases}$$

Moreover, the characteristic time for turbulent drag is

$$\begin{aligned} t_c = (\kappa g)^{-1/2} &= \sqrt{\frac{\rho_{\text{ball}}\,R}{\rho_{\text{fluid}}\,g}} \\ &= \begin{cases} \sqrt{\frac{2\times 10^{3}\,\mathrm{kg/m^3}\times 1\times 10^{-2}\,\mathrm{m}}{1\,\mathrm{kg/m^3}\times 10\,\mathrm{m/s^2}}} \simeq 1.4\,\mathrm{s} & \text{for air} \\ \sqrt{\frac{2\times 10^{3}\,\mathrm{kg/m^3}\times 1\times 10^{-2}\,\mathrm{m}}{1\times 10^{3}\,\mathrm{kg/m^3}\times 10\,\mathrm{m/s^2}}} \simeq 0.04\,\mathrm{s} & \text{for water} \end{cases} \end{aligned}$$

As a consequence, one may safely assume that either friction may be neglected or turbulent friction must be considered. Stokes friction is always negligible for the steel ball. Details will be worked out in Problem 4.6.

adapt question

4.5 Particle suspended from a spring

There are two forces acting on a particle is suspended from a spring: the gravitational force $-m\,g$ and the spring force $-k\,z(t)$ where $z(t)$ measures the displacement of the spring from its rest position. Hence, the EOM of the particle takes the form

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) \tag{4.5.1}$$

This equation can neither be integrated directly, because its right hand side depends on $z(t)$, nor can it be solved by separation of variables, because its right hand side depends on $z(t)$ rather than only on $\dot{z}(t)$. It falls into the very important class of *linear ODEs*, i.e., ODEs where $z(t)$ and its derivatives only appear as linear terms. Denoting the ν^{th} time derivative of $z(t)$ as $z^{(\nu)}(t)$, with $z^{(0)}(t) = z(t)$, an N^{th} order linear ODEs for $z(t)$ takes the general form

$$I(t) = c_N(t)\,z^{(N)}(t) + c_{N-1}(t)\,z^{(N-1)}(t) + c_{N-2}(t)\,z^{(N-2)}(t) + \cdots + c_0(t)\,z(t)$$

The functions $I(t)$, $c_\nu(t)$, $\nu = 0 \cdots N-1$, are called the coefficients of the linear ODE. When they do not depend on time we speak of a linear ODE with *constant coefficients*. In particular, $I(t)$ is called *inhomogeneity*; when it vanishes the ODE is called *homogeneous*.

Hence, Equation (4.5.1) is a second-order linear ODE with the constant coefficients $I = m\,g$, $f_0 = k$, and $f_1 = 0$.

Linear ODEs with constant coefficients are solved as follows

Algorithm 4.3: Linear ODEs with constant coefficients

An N^{th}-order linear ODE with constant coefficients,

$$I = \sum_{\nu=0}^{N} c_\nu \, f^{(\nu)}(t)$$

can be recast into a homogeneous ODE by considering $h(t) = f(t) - I/c_0$, which is a solution of the corresponding homogeneous, linear ODE

$$0 = \sum_{\nu=0}^{N} c_\nu \, h^{(\nu)}(t)$$

Its solutions can be written as

$$h(t) = \sum_{k=1}^{N} A_k \, \mathrm{e}^{\lambda_k t}$$

where the numbers λ_k, $k = 1 \cdots N$ are the roots of the characteristic polynomial

$$0 = \sum_{\nu=0}^{N} c_\nu \, \lambda^\nu$$

and the amplitudes A_k, $k = 1 \ldots N$ must be chosen such that $f(t) = I + c_0 \, h(t)$ obeys the initial conditions

$$\begin{aligned} f(t_0) &= \frac{I}{c_0} + \sum_{k=1}^{N} A_k \, \mathrm{e}^{\lambda_k t_0} \\ f^{(1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k \, \mathrm{e}^{\lambda_k t_0} \\ \vdots &= \vdots \\ f^{(N-1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k^{N-1} \, \mathrm{e}^{\lambda_k t_0} \end{aligned}$$

For Equation (4.5.1) this implies that $h(t) = z(t) + mg/k$ with

$$0 = \ddot{h}(t) + \frac{k}{m} \, h(t)$$

such that we obtain

$$\lambda_\pm = \pm\sqrt{\frac{k}{m}} = \pm\omega \qquad \text{as solution of} \quad 0 = \lambda^2 + \frac{k}{m}$$

Consequently, the motion of the spring is described by

$$z(t) = -\frac{mg}{k} + A_+ \, \mathrm{e}^{\omega\,(t-t_0)} + A_- \, \mathrm{e}^{-\omega\,(t-t_0)}$$

This is a real-valued function if and only if A_+ and A_- are canonically conjugated complex numbers, such that we can write $A_\pm =$

$A\,\mathrm{e}^{\pm\varphi}/2$. As a consequence of $\cos x = (\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})/2$ we then obtain

$$z(t) = -\frac{mg}{k} + A\ \cos(\varphi + \omega\,(t - t_0))$$

where A and φ must be fixed based on the initial conditions

$$\begin{aligned} z(t_0) &= -\frac{mg}{k} + A\ \cos(\varphi) \\ \dot{z}(t_0) &= \qquad -\omega\ A\ \sin(\varphi) \end{aligned}$$

or

$$A^2 = \left(z(t_0) + \frac{mg}{k}\right)^2 + \frac{\dot{z}^2(t_0)}{\omega^2} \qquad \text{and} \qquad \varphi = \arcsin\left(\frac{\dot{z}(t_0)}{\omega\ A}\right)$$

add: variation of constants: fish pond, bells

4.6 *Kepler's laws for planetary motion — a worked example*

add worked axample

4.7 *Problems*

Rehearsing Concepts

Problem 4.1. **Stokes friction.**

The EOM for Stokes friction, Equation (4.3.1) is a linear differential equation. Adopt the strategy for solving linear differential equations, Algorithm 4.3, to find the solution Equation (4.3.5).

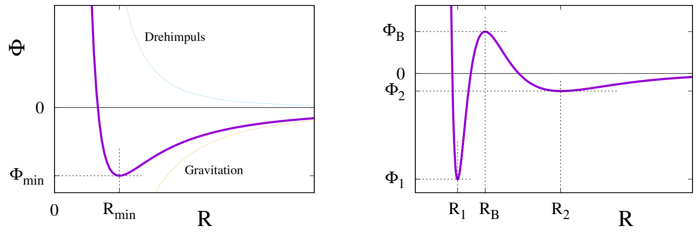
Problem 4.2. **Phase-space portraits for a scattering problem.**

Sketch the potential $\Phi(x) = 1 - 1/\cosh x$ for $x \in \mathbb{R}$. Add to the sketch a the phase portrait of the motion in this potential, i.e. the solutions of $\ddot{x} = -\partial_x \Phi(x)$, in the phase space $(x, \dot{x})$.

Problem 4.3. **Phase-space portraits for the Kepler and the DLVO problem.**

The figures below show the effective potentials for the distance between two planets in the Kepler problem, and for the DLVO potential for the interaction of charged colloids.[1] Sketch the solutions for classical trajectories in these potentials in the phase space $(R, \dot{R})$.

[1] The DLVO theory predicts that there are two distinct average bond length for aggregates of two colloids. There is a strong bond of strength Φ_1 where the colloids have a small bond length R_1, and a weak bond of strength Φ_2 at a larger distance R_2. Between these two states there is an energy barrier of height Φ_B.



Problem 4.4. **Gradients and equipotential lines**

Determine the derivatives of the following functions.

a) Equipotential lines in the (x, y)-plane are lines $y(x)$ or $y(x)$ where a functions $f(x, y)$ takes a constant value. Sketch the equipotential lines of the functions

$$f_1(x, y) = (x^2 + y^2)^{-1} \qquad \text{and} \qquad f_2(x, y) = -x^2\, y^2$$

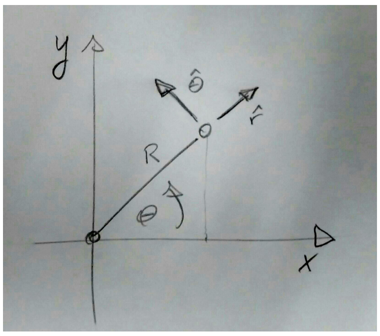
b) Determine the gradients $\nabla f_1(x, y)$ and $\nabla f_2(x, y)$.
Hint: The gradient $\nabla f_i(x, y)$ with $i \in \{1, 2\}$ is a vector $(\partial_x f_i(x, y), \partial_y f_i(x, y))$ that contains the two partial derivatives of the (scalar) function $f_i(x, y)$.

c) Indicate the direction and magnitude of the gradient by appropriate arrows in the sketch showing the equipotential lines. In which direction is the gradient pointing?

Practicing Concepts

Problem 4.5. **Motion on a circular track.**

The position of a particle in the plane can be specified by Cartesian coordinates (x, y) of by polar coordinates with basis vectors $\hat{r}(\theta)$ and $\hat{\theta}(\theta)$, that have the following representation in Cartesian coordinates (cf. the sketch to the left)



$$\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

We will now explore the trajectory $\vec{q}(t)$ of a particle with mass m that moves on a track with a fixed radius R.

a) Verify that $\hat{\theta} = \partial_\theta \hat{r}$ and $\partial_\theta^2 \hat{r} = -\hat{r}$.
Please provide a geometric interpretation of this result!

b) The position of the particle can be specified as $\vec{q}(t) = R\,\hat{r}(\theta(t))$. Determine $\dot{\vec{q}}$ and $\ddot{\vec{q}}$ based on this equations. Verify your result by performing the same calculation in Cartesian coordinates.

c) Which force is required to keep the particle on the circular track? What does this imply for curves in bike races, bobsled races and in skate parks?

d) Consider the motion at a constant angular velocity, $\theta(t) = \omega t$, and show that the acceleration in this setting takes the form $\ddot{\vec{q}} = -R\,\omega^2\,\hat{r}(\omega t)$ Verify that this amounts to a force that is perpendicular to the velocity. What does this imply for the absolute value of the velocity?
Hint: Discuss the time derivative of $\vec{v}^2$.

adapt question

Problem 4.6. **Free fall with viscous friction.**

The falling of a ball in a viscous medium can be described by the equations of motion

$$\ddot{h}(t) = -g - \gamma\,\dot{h}(t)\,.$$

Here $h(t)$ is the vertical position of the ball, g is the acceleration due to gravity, and the coefficient $\gamma \simeq \frac{2}{3}\eta/\rho_{\text{ball}}R^2$ describes the viscous drag. Here R is the radius of the sphere, ρ_{ball} is the mass density of its material, and η is the viscosity of the surrounding fluid. For air and water it takes values of about $\eta_{\text{air}} \simeq 2 \times 10^{-5}\,\text{kg/m s}$, and $\eta_{\text{water}} \simeq 1 \times 10^{-3}\,\text{kg/m s}$, respectively.

a) Argue that $w(\tau) = \gamma\dot{h}(t)/g$ with $\tau = \gamma t$ obeys the equation

$$\frac{\mathrm{d}w(\tau)}{\mathrm{d}\tau} = -1 - w(\tau)\,.$$

How do you recover the the dependence of the motion on the parameters g and γ?

b) Determine the solution of the equation for the initial condition $w(\tau_0) = w_0$. What happens for $w_0 = -1$?

c) Determine $h(t)$ for a ball that is released at a height H with zero velocity, and with an upward velocity of v_0.

d) Sketch the solution $h(t)$. How does the solution look like for small and for large t?
In particular: Determine the Taylor expansion for the trajectory to third order in t. How does it differ from a free fall with $\gamma = 0$?

e) Estimate the time scale where the viscosity does not yet lead to noticeable differences from a description with viscosity of a bullet with a diameter of 1 cm, when it drops down from the balcony and when it is vertically shot into the air with initial velocity 100 m/s. How far did it travel in that time?

f) How do the conclusions change for a harpoon shot under water? (Assume for simplicity that it is sufficient to treat it like a ball with radius corresponding to the diameter of the arrow.)

Problem 4.7. **Damped oscillator.**

Physical systems are subjected to friction. This can be taken into account by augmenting the EOM of a particle suspended from a spring, Equation (4.5.1), by a friction term

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) - \mu\,\dot{z}(t)$$

a) How does friction affect the motion $z(t)$ of the particle? What is the condition that there are still oscillations, even though with a damping? For which parameters will they disappear, and how do the solutions look like in that case?

b) Sketch the evolution of the trajectories in phase space, for the two settings with and without oscillations.

c) For the borderline case the characteristic polynomial will only have a single root, λ. Verify that the general solution can then be written as

$$z(t) = A_1\,\mathrm{e}^{\lambda\,(t-t_0)} + A_2\,t\,\mathrm{e}^{\lambda\,(t-t_0)}$$

d) Determine the solutions for a particle for the following initial conditions:
the particle is at rest and at a distance A from its equilibrium position,
the particle is at the equilibrium position, but it has an initial velocity v_0.
Indicate the form of these trajectories in the phase-space plots.

Proofs

Transfer and Bonus Problems, Riddles

Problem 4.8. **Maximum distance of flight.**

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

Challenge. What is the optimum choice of θ for the shot put?

c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

Take Home Message
Hints for Exam Preparation

The aim of the present course has been to give a first gimps into scientific modeling. It focussed on mechanics problems. Firstly, they are easily visualized. Secondly, they readily provide interesting mathematical challenges when one strives for a comprehensive description. Thus, they are provide a unique set of problems to get acquainted with the use of mathematics as a language to address scientific problems. The involved mathematical concepts will further be underpinned in forthcoming mathematics classes. Further physical problems will be addressed in experimental and theoretical physics lectures.

What are the next steps to be taken? To begin with you should re-read the script and revisit the exercise sheets in order to prepare for the exam. Take a particular look at exercises that were challenging at the first encounter. In doing so you should focus on understanding the rules of the game, and hands-on application of the mathematical formalism, rather than understanding the concepts in full depth. The concepts will be dealt with again in mathematics classese, that will *not* address, however, practicalities about efficient use in concrete calculations.

Best wishes, success and fun for your further studies!

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Todo list

The Todo List is empty!
Why?