

Lecture Notes by Jürgen Vollmer

# Theoretical Mechanics

— Working Copy —

— 2020-01-08 14:29:12+01:00—

Copyright © 2019 Jürgen Vollmer

LECTURES DELIVERED AT FAKULTÄT FÜR PHYSIK UND GEOWISSENSCHAFTEN, UNIVERSITÄT LEIPZIG

<http://www.uni-leipzig.de/physik/~vollmer>

Die Philosophie steht in diesem großen Buch geschrieben, dem Universum, das unserem Blick ständig offen liegt. Aber das Buch ist nicht zu verstehen, wenn man nicht zuvor die Sprache erlernt und sich mit den Buchstaben vertraut gemacht hat, in denen es geschrieben ist. Es ist in der Sprache der Mathematik geschrieben, und deren Buchstaben sind Kreise, Dreiecke und andere geometrische Figuren, ohne die es dem Menschen unmöglich ist, ein einziges Wort davon zu verstehen; ohne diese irrt man in einem dunklen Labyrinth herum.

GALILEO GALILEI, *Il Saggiatore*, 1623

Die Mathematik ist das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten: sie baut die verbindende Brücke und gestaltet sie immer tragfähiger. Daher kommt es, daß unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Natur beruht, ihre Grundlage in der Mathematik findet.

DAVID HILBERT, Ansprache "Naturerkennen und Logik" am 8.9.1930 während des Kongresses der Vereinigung deutscher Naturwissenschaftler und Mediziner

Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.

ALBERT EINSTEIN Festvortrag "Geometrie und Erfahrung" am 27.1.1921 vor der Preußischen Akademie der Wissenschaften



# Preface

*Die ganzen Zahlen hat der liebe Gott geschaffen,  
alles andere ist Menschenwerk.*

Leopold KRONECKER

Almost 400 years ago GALILEI GALILEO expressed the credo of modern sciences: The language of mathematics is the appropriate instrument to decode the secrets of the universe. Arguably the fruits of this enterprise are more visible today than they have ever been in the past. Mathematical models are the cornerstone of modern science and engineering. They provide the tools for optimizing engines, and the technology for data and communication sciences. No car will run, no plane will fly, no cell phone ring without the technical equipment and the software to make it run. Moreover, ever again the challenges of physics models inspired the development of new mathematics. Indeed, physics and mathematics take complementary perspectives: Mathematicians strive for a logically stringent representation of the structure of theories and models. Physicists adopt mathematics as a tool to speak about and better understand nature.

The present Lecture Notes accompany the course “Theoretical Mechanics” for physics freshmen in the international physics program of the [Universität Leipzig](#). The course addresses mechanics problems to introduce the students to concepts and strategies aiming at a quantitative description of observations. Hence, the lectures have several purposes:

- a) They introduce the concept of a mathematical model, its predictions, and how they relate to observations.
- b) They present strategies adopted to develop a model, to explore its predictions, to falsify models, and to refine them based on comparison to observations.
- c) They introduce mathematical concepts used in this enterprise:

dimensional analysis, non-dimensionalization, complex numbers, vector calculus, and ordinary differential equations.

- d) They provide an introduction to Newtonian and Lagrangian Mechanics.

Our approach to mathematical concepts is strongly biased to developing the skills to apply the tools in a modeling context, rather than striving for mathematical rigor. For the latter we point out potential pitfalls based on physical examples, and refer the students to forthcoming maths classes. On request and/or need additional topics, that are useful as a reference and to rehearse elementary concepts, can be added in appendices or outlook sections in the chapters.

The material is organized in chapters that address subsequent mathematical and physical topics. Each chapter is introduced by a physics illustration problem. Then, we develop and discuss relevant new concepts. Subsequently, we provide a worked examples. One of them will be the solution of the problem sketched in the introduction. Finally, there is a section with different types of problems:

- a. quickies to test conceptual understanding and highlight the new concepts.
- b. exercises to gain practice in employing the concepts.
- c. more elaborate exercises where the new concepts are used to discuss non-trivial problems.

d. exercises that provide complementary insight based on Python and Sage programs

e. teasers with challenging problems. Typically these exercises require a non-trivial combination of different concepts that have been introduced in earlier chapters.

At the end of the chapter we recommend additional literature and provide an outlook for further reading.

I am eager to receive feedback. It will be crucial for the development of this project to learn about typos, inconsistencies, confusing or incomplete explanations, and suggestions for additional material (contents as well as links to papers, books and internet resources) that should be added in forthcoming revisions. Everybody who is willing to provide feedback will be invited to a coffee in Cafe Corso.

add reading plan

add more explanation

acknowledge co-workers

# *Contents*

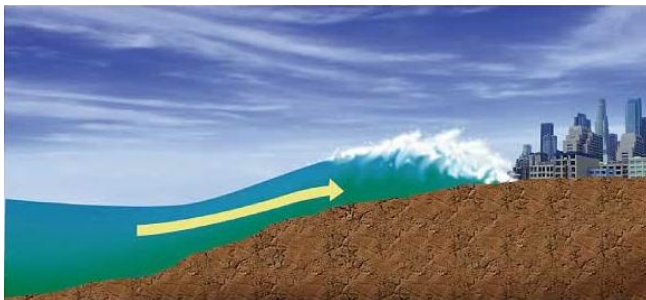
<b>Preface</b>	<b>v</b>
<b>1 Basic Principles</b>	<b>1</b>
<b>2 Balancing forces and torques</b>	<b>13</b>
<b>3 Newton's Laws</b>	<b>37</b>
<b>4 Equations of Motion</b>	<b>55</b>
<b>Take Home Messags</b>	<b>65</b>
<b>Todo list</b>	<b>66</b>





# 1

## Basic Principles



[-Ilhador- Public domain]

At the end of this chapter we will be able to estimate the speed of a Tsunami wave.

### 1.1 Basic Notions of Mechanics

#### Definition 1.1: System

A mechanical *system* is comprised of a set of masses  $m_i$ ,  $i \in \mathbb{I}$ , at positions  $\vec{x}_i$  that move with velocities  $\vec{v}_i$ .

*Remark 1.1.* We say that the system has  $N$  particles when  $\mathbb{I} = \{1, \dots, N\}$ .

*Remark 1.2.* The arrows indicate here that  $\vec{x}_i$  describes a position in space. For a  $D$  space  $\vec{x}_i \in \mathbb{R}^D$ , and we say that  $\vec{x}_i$  is a  $D$ -vector.

*Remark 1.3.* In order to emphasize the close connection between positions and velocities, the latter will also be denoted as  $\dot{\vec{x}}$ .

**Example 1.1: A piece of chalk**

We wish to follow the trajectory of a piece of chalk through the lecture hall. In order to follow its position and orientation in space, we *decide* to model it as a set of two masses that are localized at the tip and at the tail of the chalk. The positions of these two masses  $\vec{x}_1$  and  $\vec{x}_2$  will both be vectors in  $\mathbb{R}^3$ . For instance we can indicate the shortest distance to three walls that meet in one corner of the lecture hall. In this model we have  $N = 2$  and  $D = 3$ .

**Definition 1.2: Degrees of Freedom (DOF)**

A system with  $N$  particles whose positions are described by  $D$  vectors has  $DN$  degrees of freedom (DOF).

*Remark 1.4.* Note that according to this definition the number of DOF is a property of the model. For instance, the model for the piece of chalk has  $DN = 6$  DOF. However, the length of the piece of chalk does not change. Therefore, one can find an alternative description that will only evolve 5 DOF. (We will come back to this in due time.)

**Definition 1.3: State Vector**

The position of all particles can be written in a single *state vector*,  $\vec{q}$ , that specifies the positions of all particles.

*Remark 1.5.* For a system with  $N$  particles whose positions are specified by  $D$ -dimensional vectors,  $\vec{x}_i = (x_{i,1}, \dots, x_{i,D})$ , the vector  $\vec{q}$  takes the form  $\vec{q} = (x_{1,1}, \dots, x_{1,D}, x_{2,1}, \dots, x_{2,D}, \dots, x_{N,1}, \dots, x_{N,D})$ . For conciseness we will also write  $\vec{q} = (\vec{x}_1, \dots, \vec{x}_N)$ . The vector  $\vec{q}$  has  $DN$  number of entries.

*Remark 1.6.* The velocity associated to  $\vec{q}$  will be denoted as  $\dot{\vec{q}} = (\dot{\vec{x}}_1, \dots, \dot{\vec{x}}_N)$ .

**Definition 1.4: Phase Vector**

The position and velocities of all particles form the *phase vector*,  $\vec{\Gamma} = (\vec{q}, \dot{\vec{q}})$ .

**Definition 1.5: Trajectory**

The *trajectory* of a system is described by specifying the time dependent functions

$$\begin{aligned} & \vec{x}_i(t), \vec{v}_i(t), \quad i = 1, \dots, N \\ \text{or } & \vec{q}(t), \dot{\vec{q}}(t) \\ \text{or } & \vec{\Gamma}(t) \end{aligned}$$

**Definition 1.6: Initial Conditions (IC)**

For  $t \in [t_0, \infty)$  the trajectory is uniquely determined by its *initial conditions (IC)* for the positions  $\vec{x}_i(t_0)$  and velocities  $\vec{v}_i(t_0)$ , i.e. the point  $\vec{\Gamma}(t_0)$  in phase space.

*Remark 1.7.* This definition expresses that the future evolution of a system is *uniquely* determined by its ICs. Such a system is called deterministic. Mechanics addresses the evolution of deterministic systems. At some point in your studies you might encounter stochastic dynamics where different rules apply.

**Example 1.2: Throwing a javelin**

The ICs for the flight of a javelin specify where it is released,  $\vec{x}$ , when it is thrown, the velocity  $\vec{v}$  at that point of time, and the orientation of the javelin. In a good trial the initial orientation is parallel to  $\vec{v}$ , as shown in Figure 1.1

*Remark 1.8.* In repeated experiments the ICs will be (slightly) different, and one observes different trajectories.

1. A seasoned soccer player will hit the goal in repeated kicks. However, even a professional may miss occasionally.

2. A bicycle involves a lot of mechanical pieces that work together to provide a predictable riding experience.

3. A lottery machine involves a smaller set of pieces where a different (in practice unpredictable) set of balls is selected in each run, in spite of best efforts to select identical initial conditions.

**Definition 1.7: Constant of Motion**

A function of the positions  $\vec{x}_i$  and velocities  $\vec{v}_i$  that does not evolve in time is called a *constant of motion*.



Figure 1.1: Initial conditions for throwing a javelin, cf. Example 1.2. [Atalanta, creativecommons, CC BY-SA 3.0]

**Example 1.3: A piece of chalk**

During the flight the positions  $\vec{x}_1$  and  $\vec{x}_2$  of the piece of chalk will change. However, the length  $L$  of the piece of chalk will not, and at any given time it can be determined from  $\vec{x}_1$  and  $\vec{x}_2$ . Hence,  $L$  is a constant of motion.

**Definition 1.8: Parameter**

In addition to the ICs the trajectories will depend on *parameters* of the system. Their values are fixed for a given system.

**Example 1.4: A piece of chalk**

For the piece of chalk the trajectory will depend on whether the hall is the Theory Lecture Hall in Leipzig, a briefing room in a ship during a heavy storm, or the experimental hall of the ISS space station. To the very least one must specify how the gravitational acceleration acts on the piece of chalk, and how the room moves in space.

*Remark 1.9.* The set of parameters that appear in a model depends on the *choices* that one makes upon setting up the experiment. For instance

Beckham's kicks can only be understood when one accounts for the impact of air friction on the soccer ball.

Air friction will not impact the trajectory of a small piece of talk that I through into the dust bin.

By adopting a clever choice of the parameterization the trajectory of the piece of chalk can be described in a setting with 4 DOF. The length of the piece of chalk will appear as a parameter in that description.

**Definition 1.9: Physical Quantities**

Positions, velocities and parameters are *physical quantities* that are characterized by at least one numbers and a unit.

**Example 1.5: Physical Quantities**

1. The mass of a soccer ball can be fully characterized by a number and the unit kilogram (kg), e.g.  $M \approx 0.8 \text{ kg}$ .
2. The length of a piece of chalk can be fully characterized by a number and the unit meter (m), e.g.  $L \approx 7 \times 10^{-2} \text{ m}$ .
3. The length  $T$  of a year can be characterized by a number and the unit second, e.g.  $T \approx \pi \times 10^7 \text{ s}$ .
4. The speed of a car can be fully characterized by a number and the unit, e.g.  $v \approx 72 \text{ km h}^{-1}$ .
5. A position in a  $D$ -dimensional space can fully be characterized by  $D$  numbers and the unit meter.
6. The velocity of a piece of chalk flying through the lecture hall can be characterized by three numbers and the unit m/s.

*Remark 1.10.* Analyzing the units of the parameters of a system provides a fast way to explore and write down functional dependencies. When doing so, the units of a physical quantity  $Q$  are denoted by  $[Q]$ . For instance for the length  $L$  of the piece of chalk, we have  $[L] = \text{m}$ . For a dimensionless quantity  $d$  we write  $[d] = 1$ .

**Example 1.6: Changing units**

Suppose we wish to change units from km/h to m/s. A transparent way to do this for the speed of the car in the example above is by multiplications with one

$$v = 72 \frac{\text{km}}{\text{h}} \frac{1 \text{ h}}{3.6 \times 10^3 \text{ s}} \frac{1 \times 10^3 \text{ m}}{1 \text{ km}} = \frac{72}{3.6} \text{ m s}^{-1} = 20 \text{ m s}^{-1}$$

**Definition 1.10: Dynamics**

The characterization of all possible trajectories for all admissible ICs is called *dynamics* of a system.

## 1.2 Dimensional Analysis

Mathematics does not know units. Experimental physicists hate large sets of parameters because the sampling of high-dimensional parameter space is tiresome. A remedy to both issues is offered by the Buckingham-Pi-Theorem. We state it here in a form accessible with our present level of mathematical refinement. The discussion

of a more advanced formulation may appear as a homework problem later on on this course.

### Theorem 1.1: Buckingham-Pi-Theorem

A dynamics with  $n$  parameters, where the positions  $\vec{q}$  and the parameters involve the three units meter, seconds and kilogram, can be rewritten in terms of a *dimensionless dynamics* with  $n - 3$  parameters, where the positions  $\vec{\zeta}$ , velocities  $\vec{\zeta}$ , and parameters  $\pi_j$  with  $j \in \{1, \dots, n - 1\}$  are given solely by numbers.

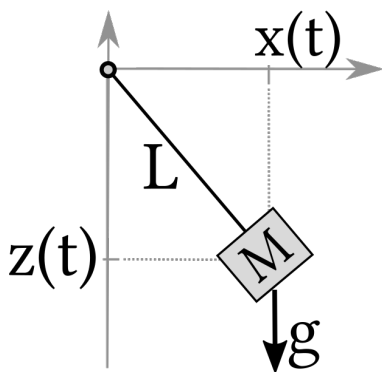


Figure 1.2: Pendulum discussed in Example 1.7

### Example 1.7: Non-dimensionalization for a pendulum

Let  $\vec{x}$  denote the position of a pendulum of mass  $M$  that is attached to a chord of length  $L$  and swinging in a gravitational field  $\vec{g}$  of strength  $g$  (see Figure 1.2).

The units of these quantities are  $[\vec{x}] = \text{m}$ ,  $[M] = \text{kg}$ ,  $L = \text{m}$ , and  $[g] = \text{m/s}^2$ , respectively. There are three parameters,  $n = 3$ , plus the direction of  $\vec{g}$ .

In this problem we choose  $L$  as length scale and  $\sqrt{L/g}$  as time scale. Then the dimensionless positions will be  $\vec{\zeta} = \vec{q}/L$ , the dimensionless velocities will be  $\vec{\zeta} = \vec{q}/\sqrt{gL}$ .

There is no way to turn  $M$  into a dimensionless parameter. Therefore the evolution of  $(\zeta, \zeta)$  can not depend on  $M$ . The only dimensionless parameter that remains in the model is the direction of  $\vec{g}$ .

**Example 1.8: Non-dimensionalization for the flight of a piece of chalk**

Let  $\vec{x}_1$  and  $\vec{x}_2$  denote the position of the tip and the tail of a model for a piece of chalk, where tip and tail are associated to masses  $m_1$  and  $m_2$ . The piece of chalk has a length  $L$ . It performs a free flight in a gravitational field with acceleration  $\vec{g}$  of strength  $g$ .

The units of these quantities are  $[\vec{x}_i] = \text{m}$ ,  $[m_i] = \text{kg}$ ,  $L = \text{m}$ , and  $[g] = \text{m/s}^2$ , respectively. There are four parameters,  $n = 4$ , plus the direction of  $\vec{g}$ .

In this problem we choose  $L$  as length scale and  $\sqrt{L/g}$  as time scale. Then the dimensionless positions will be  $\vec{\zeta} = \vec{q}/L$ , the dimensionless velocities will be  $\vec{\zeta} = \vec{q}/\sqrt{gL}$ .

The two masses  $m_1$  and  $m_2$  give rise to the dimensionless parameter  $\pi_1 = m_1/m_2$ , and in three dimensions the direction of  $\vec{g}$  must be characterized by another two dimensionless parameters.

*Proof.* We first look for combinations of the parameters with the following units

$$\begin{aligned} \text{m} &= [p_1^{\alpha_1}] [p_2^{\alpha_2}] \dots [p_n^{\alpha_n}] \\ \text{s} &= [p_1^{\beta_1}] [p_2^{\beta_2}] \dots [p_n^{\beta_n}] \\ \text{kg} &= [p_1^{\gamma_1}] [p_2^{\gamma_2}] \dots [p_n^{\gamma_n}] \end{aligned}$$

Each of these equations involves constraints on the exponents in order to match the exponents of the three units that can be expressed as a system of linear equations. The solvability conditions for such systems imply that they conditions can always be met by an appropriately chosen set of three parameters. Without loss of generality we denote them as  $p_1$ ,  $p_2$  and  $p_3$ , and we have

$$\begin{aligned} \text{m} &= [p_1^{\alpha_1}] [p_2^{\alpha_2}] [p_3^{\alpha_n}] \\ \text{s} &= [p_1^{\beta_1}] [p_2^{\beta_2}] [p_3^{\beta_n}] \\ \text{kg} &= [p_1^{\gamma_1}] [p_2^{\gamma_2}] [p_3^{\gamma_n}] \end{aligned} \tag{1.2.1}$$

Thus we use the parameters  $p_1, \dots, p_3$  to remove the units from our

description. In its dimensionless form it will involve the positions and velocities

$$\vec{\xi} = \vec{q} p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_n}$$

$$\vec{\zeta} = \dot{\vec{q}} p_1^{\beta_1 - \alpha_1} p_2^{\beta_2 - \alpha_2} p_3^{\beta_n - \alpha_n}$$

Similarly, the dimensionless form of the parameters  $p_i$  of the dynamics are obtained by multiplying the original parameters with appropriate powers of the expressions (1.2.1) of the units. For  $p_1$  to  $p_3$  this gives rise to one. Additional parameters will turn into dimensionless groups of parameters  $\pi_1$  to  $\pi_{n-3}$ .  $\square$

### 1.3 Order-of-magnitude guesses

Many physical quantities take a value close to one when they are expressed in their “natural” dimensionless units. When the choice is unique, then clearly it is also natural. Otherwise, the appropriate choice is a matter of experience.

We will come back to this when we employ non-dimensionalization in the forthcoming discussion. We demonstrate this based on a discussion of

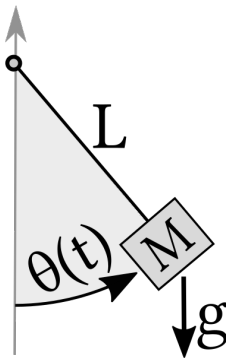


Figure 1.3: Pendulum discussed in Example 1.9

#### Example 1.9: The period of a pendulum

We consider a pendulum of mass  $M$  attached at a stiff bar of negligible mass. With this bar it is fixed to a pivot at a distance  $L$  from the mass such that it can swing in a gravitational field inducing an acceleration  $g$  (see Figure 1.3).

As discussed in Example 1.7 the dimensionless time unit for this problem is  $\sqrt{L/g}$ . Hence we estimate that the period  $T$  of the pendulum is  $T \simeq \sqrt{L/g}$ . Explicit calculations to be performed later on will reveal that this estimate is off by a factor  $2\pi$  when the period is small. For large oscillation amplitudes  $\theta_0$  the period will increase further, tending to infinity when  $\theta_0$  approaches  $\pi$ . Hence, we conclude that

$$T = f(\theta_0) \sqrt{L/g} \quad \text{with } f(\theta_0) \simeq 2\pi \text{ for } \theta_0 \ll 1.$$



**Example 1.10: The speed of Tsunami waves**

A Tsunami wave is a water wave that is generated by an earth quake or an underwater land slide. Typical wave lengths are of an order of magnitude  $\lambda = 100$  km. They travel through the ocean that has an average depth of about  $D = 4$  km, much smaller than  $\lambda$ . Therefore, we expect that the wave speed  $v_{\text{Tsunami}}$  is predominantly set by the ocean depth and the gravitational acceleration  $g \approx 10 \text{ m/s}^2$ , i.e.

$$v_{\text{Tsunami}} \approx \sqrt{gD} = 2 \times 10^2 \text{ m/s} \approx 700 \text{ km/h}$$

This estimate is very close to the observed values. Hence, the 2004 Indian Ocean Tsunami traversed the distance from Indonesia to the East African coast,  $L \approx 10\,000$  km in only

$$\frac{L}{v_{\text{Tsunami}}} \approx \frac{1 \times 10^4 \text{ km}}{700 \text{ km/h}} = \frac{100}{7} \text{ h} \approx 15 \text{ h}$$

It was 16 h according to [Wikipedia](#). However, in spite of their speed and devastating power, Tsunamis are very hard to detect on the open sea because their period  $T$  is very long. It can be estimated as the time that the wave needs to run once through its wavelength<sup>1</sup>

$$T \approx \frac{\lambda}{v_{\text{Tsunami}}} = \frac{\lambda}{\sqrt{gD}} = \frac{100 \text{ km}}{700 \text{ km/h}} = \frac{1}{7} \text{ h} \approx 10 \text{ min}$$

Here, our estimate is too small by about a factor of three.

<sup>1</sup> Observe that this physical argument goes beyond the blind use of dimensional analysis. The equation for  $T$  involves the length scales  $\lambda$  and  $D$  in a non-trivial combination that is set by a physical argument.

We conclude that estimates based on dimensional analysis provide valuable insight in time scales of physical processes, even in situations where a detailed mathematical treatment is very delicate.

## 1.4 Problems

*Rehearsing Concepts**Problem 1.1. Printing the output of Phantom cameras*

With a set of three phantom cameras one can simultaneously follow the motion of 100 particles in a violent 3d turbulent flow. Data analysis of the images provides particle positions with a resolution of 25,000 frames per second. You follow the evolution for 20 minute, print it double paged with 8 coordinates per line and 70 lines per

page. A bookbinder makes 12 cm thick books from every 1000 pages. You put these books into bookshelves with seven boards in each shelf. How many meters of bookshelves will you need to store your data on paper?

### *Practicing Concepts*

*Problem 1.2.* Oscillation Period of a Particle attached to a spring  
In a gravitational field with acceleration  $g_{\text{Moon}} = 1.6 \text{ m/s}^2$  a particle of mass  $M = 100 \text{ g}$  is hanging at a spring with spring constant  $k = 1.6 \text{ kg/s}^2$ . It oscillates with period  $T$  when it is slightly pulled downwards and released. We describe the oscillation by the distance  $x(t)$  from its rest position.

- Determine the dimensionless distance  $\zeta(t)$ ,  
and the associated dimensionless velocity  $\dot{\zeta}(t)$ .
- Provide an order-of-estimate guess of the oscillation period  $T$ .

*Problem 1.3.* Earth orbit around the sun

- Light travels with a speed of  $c \approx 3 \times 10^8 \text{ m s}^{-1}$ . It takes 8 minutes and 19 seconds to travel from Sun to Earth. What is the distance  $D$  of Earth and Sun in meters?
- The period of the trajectory of the Earth around the Sun depends on  $D$ , on the mass  $M = 2 \times 10^{30} \text{ kg}$  of the sun, and on the gravitational constant  $G = 6.7 \times 10^{-11} \text{ m}^3/\text{kg s}^2$ . Estimate, based on this information, how long it takes for the Earth to travel once around the sun.
- Express your estimate in terms of years. The estimate of (b) is of order one, but still off by a considerable factor. Do you recognize the numerical value of this factor?
- Upon discussing the trajectory  $\vec{x}(t)$  of planets around the sun later on in this course, we will introduce dimensionless positions of the planets  $\vec{\zeta}(t) = \vec{x}(t)/L = (x_1(t)/L, x_2(t)/L, x_3(t)/L)$ . How would you define the associated dimensionless velocities?

*Problem 1.4.* Water Waves

The speed of waves on the ocean depends only on their wave length  $L$  and the gravitational acceleration  $g \simeq 10 \text{ m/s}^2$ .

- How does the speed of the waves depend on  $L$  and  $g$ ?

- b) Unless it is surfing, the speed of a yacht is limited by its hull speed, i.e. the speed of a wave with wave length identical to the length of the yacht. Estimate the top speed of a 30 ft yacht.
- c) Close to the beach the water depth  $H$  become a more important parameter than the wave length. How does the speed of the crest and the trough of the wave differ? What does this imply about the form of the wave?

*Proofs*

*Transfer and Bonus Problems, Riddles*



## 2

### *Balancing forces and torques*

In Chapter 1 we observed that positions and velocities of particles are specified by indicating their unit, magnitude and directions. Hence, they are vectors. In the present chapter we learn how vectors are defined in mathematics, and how they are used and handled in physics. In order to provide a formal definition we will introduce a number of mathematical concepts, like groups, that will be revisited in forthcoming chapters. As first important application we will deal with balancing forces and torques.



Mobile (sculpture) in the style of Alexander Calder  
[wikimedia Creative Commons Attribution-Share Alike 2.0 Generic]

At the end of this chapter we will be able to determine how a mobile will be hanging from the ceiling.

### 2.1 Motivation and Outline: What is a vector?

In mechanics we use vectors to describe forces, displacements and velocities. A displacement describes the relative position of two points in space, and the velocity can be thought of as a distance divided by the time needed to go from the initial to the final point. (A mathematically more thorough definition will be given below.) For forces it is of paramount importance to indicate in which direction they are acting. Similarly, in contrast to speed, a velocity can not be specified in terms of a number with a unit, e.g.  $5 \text{ m s}^{-1}$ . By its very definition one also has to specify the direction of motion. Finally, also a displacement involves a length specification and a direction. For instance, the displacement from the lower left corner of a piece of paper to a point in the middle can either be specified in terms of the distance  $r$  of the point from the corner and the angle  $\theta$  of the line connecting the points and the lower edge of the paper (i.e. the direction of the point). Alternatively, it can be given in terms of two distances  $(x, y)$  that refer to the length  $x$  of a displacement along the edge of the paper and a displacement  $y$  in the direction vertical to the edge towards the paper. In three dimensions, one has to adopt a third direction out of the plane used for the paper, and hence three numbers, to specify a displacements—or indeed any other vector.

	displacement	velocity	force
	$\vec{x} = (x_1, x_2, x_3)$	$\vec{v} = (v_1, v_2, v_3)$	$\vec{F} = (f_1, f_2, f_3)$
unit	$[\vec{x}] = \text{m}$	$[\vec{v}] = \text{m s}^{-1}$	$[\vec{F}] = \text{kg m/s}^2$
magnitude	$ \vec{x}  = \sqrt{x_1^2 + x_2^2 + x_3^2}$	$ \vec{v}  = \sqrt{v_1^2 + v_2^2 + v_3^2}$	$ \vec{F}  = \sqrt{f_1^2 + f_2^2 + f_3^2}$
direction	$\hat{x} = \vec{x}/ \vec{x} $	$\hat{v} = \vec{v}/ \vec{v} $	$\hat{F} = \vec{F}/ \vec{F} $

A basic introduction of mechanics can be given based on this heuristic account of vectors. However, for the thorough exposition that serve as a fundament of theoretical physics a more profound mathematical understanding of vectors is crucial. Hence, a large part of this chapter will be devoted to mathematical concepts.

#### Outline

In the first part of this chapter we will introduce the mathematical notions of sets and groups that are needed to provide a mathemat-

ically sound definition of a vector space. Sets are the most fundamental structure of mathematics. It denotes a collection of elements, e.g., numbers like the digits of our number system  $\{1, 2, \dots, 9\}$  or the set of students in my class. Mathematical structures refer to sets where the elements obey certain additional properties, like in groups and vector spaces. They are expressed in terms of *operations* that take one or several elements of the set, and return a result that may or may not be part of the given set. When an operation  $f$  takes an element of a set  $A$  and returns another element of  $A$  we write  $f : A \rightarrow A$ . When an operation  $\circ$  takes two elements of a set  $A$  and returns a single element of  $A$  we write  $\circ : A \times A \rightarrow A$ . Equipped with the mathematical tool of vectors we will explore the physical concepts of forces and torques, and how they are balanced in objects that do not move.

## 2.2 Sets

In mathematics and physics we often wish to make statements about a collection of objects, numbers, or other distinct entities.

### Definition 2.1: Set

A *set* is a gathering of well-defined, distinct objects of our perception or thoughts.

An object  $a$  that is part of a set  $A$  is an *element* of  $A$ ; we write  $a \in A$ .

If a set  $M$  has a finite number  $n$  of elements we say that its *cardinality* is  $n$ . We write  $|M| = n$ .

*Remark 2.1.* Notationens and additional properties:

- a) When a set  $M$  has a finite number of elements, e.g.,  $+1$  and  $-1$ , one can specify the elements by explicitly stating the elements,  $M = \{+1, -1\}$ . The order does not play a role and it does not make a difference when elements are provided several times. In other words the set  $M$  of cardinality two can be specified by any of the following statements

$$M = \{-1, +1\} = \{+1, -1\} = \{-1, 1, 1, 1, \} = \{-1, 1, +1, -1\}$$

- b) If  $e$  is not an element of a set  $M$ , we write  $e \notin M$ . For instance  $-1 \in M$  and  $2 \notin M$ .

- c) There is only one set with no elements, i.e., with cardinality zero. It is denoted as  $\emptyset$ .

### Example 2.1: Sets

- Set of English names of month:

$$A_1 = \{\text{January, February, March, April, May, June, July, August, September, Oktober, November, December}\}$$

- Set of capitals of German states:

$$A_2 = \{\text{Berlin, Bremen, Hamburg, Stuttgart, Mainz, Wiesbaden, München, Magdeburg, Saarbrücken, Potsdam, Kiel, Hannover, Dresden, Schwerin, Düsseldorf, Erfurt}\}$$

- Set of small letters in German:

$$A_3 = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z, ä, ö, ü, ß\}$$

The cardinalities of these sets are

$$|A_1| = 12, |A_2| = 16, \text{ and } |A_3| = 30.$$

### Example 2.2: Sets of sets

A set can be an element of a set. For instance the set

$$M = \{1, 3, \{1, 2\}\}$$

has three elements 1, 3 und  $\{1, 2\}$  such that  $|M| = 3$ , and

$$1 \in M, \quad \{1, 2\} \in M, \quad 2 \notin M \quad \{1\} \notin M.$$

Often it is bulky to list all elements of a set. In obvious cases we use ellipses such as  $A_3 = \{a, b, c, \dots, z, ä, ö, ü, ß\}$  for the set given in Example 2.1. Alternatively, one can provide a set  $M$  by specifying the properties of its elements  $x$  in the following form

$$\underbrace{M}_{\text{The set } M \text{ contains}} = \underbrace{\{ \underbrace{x}_{\text{all elements}} \underbrace{:}_{\text{with :}} \underbrace{A(x)}_{\text{properties ...}} \}}_{\text{all elements}}$$

where the properties specify one of several properties of the elements. The properties are separated by commas, and must all be true for all elements of the set.



**Example 2.3: Set definition by property**

The set of digits  $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  can also be defined as follows  $D = \{x : 0 < x \leq 9, x \in \mathbb{Z}\}$ .

In order to specify the properties in a compact form we use logical junctors as short hand notation. In the present course we adopt the notations not  $\neg$ , and  $\wedge$ , or  $\vee$ , implies  $\Rightarrow$ , and is equivalent  $\Leftrightarrow$  for the relations indicated in 2.1.

$A$	0	0	1	1	
$B$	0	1	0	1	
$\neg A$	1	1	0	0	not $A$
$\neg B$	1	0	1	0	not $B$
$A \vee B$	0	1	1	1	$A$ or $B$
$A \wedge B$	0	0	0	1	$A$ and $B$
$A \Rightarrow B$	1	1	0	1	$A$ implies $B$
$A \Leftrightarrow B$	1	0	0	1	$A$ is equivalent to $B$
$A \vee \neg B$	1	0	1	1	$A$ or not $B$
$\neg A \wedge B$	0	1	0	0	not $A$ or $B$
$A \wedge \neg B$	0	0	1	0	$A$ and not $B$

Table 2.1: List of the results of different junctors acting on two statements  $A$  and  $B$ . Here 0 and 1 indicate that a statement is wrong or right, respectively. In the rightmost column we state the contents of the expression in the left column in words. The final three lines provide examples of more complicated expressions.

The definition of the digits in Example ?? entails that all elements of  $D$  are also numbers in  $\mathbb{Z}$ : we say that  $D$  is a subset of  $\mathbb{Z}$ .

**Definition 2.2: Subsets and Supersets**

The set  $M_1$  is a *subset* of  $M_2$ , if all elements of  $M_1$  are also contained in  $M_2$ . We write<sup>1</sup>  $M_1 \subseteq M_2$ . We denote  $M_2$  then as *superset* of  $M_1$ , writing  $M_2 \supseteq M_1$ .

The set  $M_1$  is a *proper subset* of  $M_2$  when at least one of its elements is not contained in  $M_2$ . In this case  $|M_1| < |M_2|$  and we write  $M_1 \subset M_2$ , or  $M_2 \supset M_1$ .

<sup>1</sup> Some authors use  $\subset$  instead of  $\subseteq$ , and  $\subsetneq$  to denote proper subsets.

**Example 2.4: Subsets**

- The set of month with names the end with "ber" is a subset of the set  $A_2$  of Example 2.1

$$\{\text{September, October, November, December}\} \subseteq A_3$$

- For the set  $M$  of Example 2.2 one has

$$\{1\} \subseteq M, \quad \{1, 3\} \subseteq M, \quad \{1, 2\} \not\subseteq M, \quad \{2, \{1, 2\}\} \not\subseteq M.$$

Note that  $\{1, 2\}$  is an elements of  $M$ . However, it is not a subset. The last two sets are no subsets because  $2 \notin M$ .

Two sets are the same when they are subsets of each other.

**Theorem 2.1: Equivalence of Sets**

Two sets  $A$  and  $B$  are *equal* or *equivalent*, if  $(A \subseteq B) \wedge (B \subseteq A)$ .

*Proof.*  $A \subseteq B$  implies that  $a \in A \Rightarrow a \in B$ .

$B \subseteq A$  implies  $b \in B \Rightarrow b \in A$ .

If  $A \subseteq B$  and  $B \subseteq A$ , then we also have  $a \in A \Leftrightarrow a \in B$ . □

The description of sets by properties of its members, Example 2.3, suggests that one will often be interested in operations on sets. For instance the odd and even numbers are subsets of the natural numbers, together they form this set, and when one removed the odd numbers from the natural numbers one is left with the even numbers. Hence, we define the following operations on sets.

**Definition 2.3: Set Operations**

For two sets  $M_1$  and  $M_2$  we define the following operations:

- *Intersection:*  $M_1 \cap M_2 = \{m : m \in M_1 \wedge m \in M_2\}$ ,
- *Union:*  $M_1 \cup M_2 = \{m : m \in M_1 \vee m \in M_2\}$ ,
- *Difference:*  $M_1 \setminus M_2 = \{m : m \in M_1 \wedge m \notin M_2\}$ ,
- The *complement* of a set  $M$  in a *universe*  $U$  is defined for subsets  $M \subseteq U$  as follows  $M^C = \{m \in U : m \notin M\}$ .
- The *cartesian product* of two sets  $M_1$  and  $M_2$  is defined as the set of ordered pairs  $(a, b)$  of elements  $a \in M_1$  and  $b \in M_2$ ,  
 $M_1 \times M_2 = \{(a, b) : a \in M_1, b \in M_2\}$ .

In the logical expressions following : the  $\in$  and  $\notin$  are evaluated with higher priority as the junktors  $\wedge$  and  $\vee$ .

**Example 2.5: Set operations for participants in my class**

Consider the set of participants  $P$  in my class. The sets of female  $F$  and male  $M$  participants of the class are proper subsets of  $P$  with an empty intersection  $F \cap M$ . The set of non-female participants is  $P \setminus F$ . The set of heterosexual couples in the class is a subset of the cartesian product  $F \times M$ . Furthermore, the union of the union of  $F \cup M$  is a proper subset of  $P$ , when there is at least one participant who is neither female nor male.

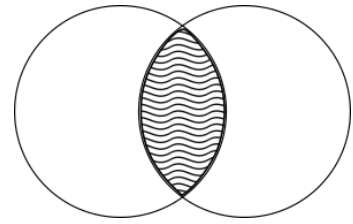


Figure 2.1: Intersection of two sets.

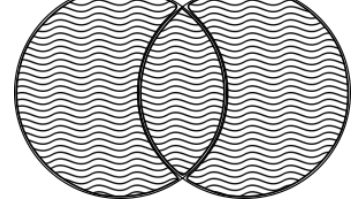


Figure 2.2: Union of two sets.

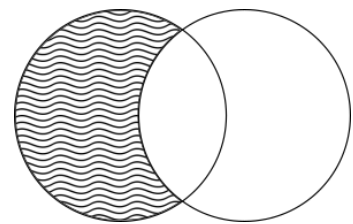


Figure 2.3: Difference of two sets.

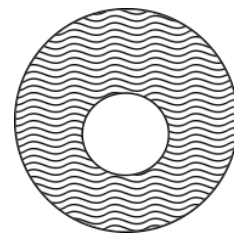


Figure 2.4: Complement of a set.

*Sets of Numbers*

Many sets of numbers that are of interest in physics have infinite many elements. We construct them in Table 2.2 based on the natural numbers

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

or the natural numbers with zero

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

*Remark 2.2.* Some authors adopt the convention that zero is in-

Table 2.2: Summary of important sets of numbers.

name	symbol	description
natural numbers	$\mathbb{N}$	$\{1, 2, 3, \dots\}$
natural numbers with 0	$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
negative numbers	$-\mathbb{N}$	$\{-n : n \in \mathbb{N}\}$
even numbers	$2\mathbb{N}$	$\{2n : n \in \mathbb{N}\}$
odd numbers	$\mathbb{Z}$	$(-\mathbb{N}) \cup \mathbb{N}_0$
rational numbers	$\mathbb{Q}$	$\left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$
real numbers	$\mathbb{R}$	see below
complex numbers	$\mathbb{C}$	$\mathbb{R} + i\mathbb{R}$ , wobei $i = \sqrt{-1}$

cluded in the natural numbers  $\mathbb{N}$ . When this matters you have to check convention is adopted.

There are many more sets of numbers. For instance, in mathematics the set of *constructable numbers* is relevant for certain proofs in geometry, and in physics we occasionally use *quaternions*. In any case one needs intervals of numbers.

#### Definition 2.4: Intervals of real numbers $\mathbb{R}$

An *interval* is a continuous subset of a set of numbers. We distinguish *open*, *closed*, and *half-open subsets*.

- closed interval:  $[a, b] = \{x : x \geq a, x \leq b\}$ ,
- open interval:  $(a, b) = ]a, b[ = \{x : x > a, x < b\}$ ,
- right open interval:  $[a, b) = [a, b[ = \{x : x \geq a, x < b\}$ ,
- left open interval:  $(a, b] = ]a, b] = \{x : x > a, x \leq b\}$ .

Subsets of  $\mathbb{R}$  will be denoted as real intervals.

### 2.3 Groups

A group refers to a set of operations that are changing some data or objects. elementary examples refer to the reflections in space, translations in space, or turning some sides of a Rubick's cube. The subsequent action of two group elements  $t_1$  and  $t_2$  will be considered to be another (typically more complicated) transformation  $t_3$ . Analogous to the concatenation of functions, we write  $t_3 = t_2 \circ t_1$ , and we say  $t_3$  is  $t_2$  after  $t_1$ . The set of transformations forms a group when one can always return to the starting point.

**Definition 2.5: Group**

A set  $(G, \circ)$  is called a *group* with operation  $\circ : G \times G \rightarrow G$  when the following rules apply

a) The set is *closed*:  $\forall g_1, g_2 \in G : g_1 \circ g_2 \in G$

b) The set has a *neutral element*:

$$\exists e \in G \forall g \in G : e \circ g = g$$

c) Each element has an *inverse element*:

$$\forall g \in G \exists i \in G : g \circ i = e$$

d) The operation is *associative*:

$$\forall g_1, g_2, g_3 \in G : (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

**Definition 2.6: Kommutative Group**

A group  $(G, \circ)$  is called a *commutative group* when

e) The operation is *commutative*:

$$\forall g_1, g_2 \in G : g_1 \circ g_2 = g_2 \circ g_1$$

**Example 2.6: Rotation-Groups are not commutative**

The rotation of an object in space is a group. In particular this holds for the  $90^\circ$ -rotations of an object around a vertical and a horizontal axis. The figures below show that these rotations do not commute:



### Example 2.7: Editing text fields

We consider the text fields of a fixed length  $n$  in an electronic form. Then the operations

“Put the letter  $\square$  into position  $\circ$  of the field”

with  $\square \in \{_, a, \dots, z, A, \dots, Z\}$

and  $\circ \in \{1, \dots, n\}$  form a group.

Also in this case one can easily check that the order of the operations is relevant. Verify that the same operations are performed in the left and right column of the following graphics:

$ -_ -_ - $	$ -_ -_ - $
$\rightarrow  M -_ -_ $	$\rightarrow  P -_ -_ $
$\rightarrow  M a -_ $	$\rightarrow  P h -_ $
$\rightarrow  M a t -_ $	$\rightarrow  P h y -_ $
$\rightarrow  M a t h $	$\rightarrow  P h y s $
$\rightarrow  M a t s $	$\rightarrow  P h y h $
$\rightarrow  M a y s $	$\rightarrow  P h t h $
$\rightarrow  M h y s $	$\rightarrow  P a t h $
$\rightarrow  P h y s $	$\rightarrow  M a t h $

*Remark 2.3.* Notationens and additional properties:

- a) Depending of the context the inverse element will be denoted as  $g^{-1}$  oder als  $-g$  notiert. This will depend on whether the operation is considered a multiplication or rather an addition. In accordance with this choice the netral element will be denoted as 1 or 0.
- b) the requirement (b)  $e \circ g = g$  implies that also  $g \circ e = g$ . The proof is provided as exercise ??.
- c) When a group is not abelian then one must distinguish the left and right inverse. The condition  $g \circ i = e$  does not imply  $i \circ g = e$ . However, there always will be another element  $j \in G$  such that  $j \circ g = e$ .

The empty set can not be a group because it has no neutral element. Therefore the smallest groups has a single element

**Example 2.8: The smallest group**

$(\{n\}, \circ)$  comprises only the neutral element.

*Sets of Numbers*

Besides being of importance to characterize the action of discrete symmetry operations like reflections or rotations by fixed angles, groups are also important for us because they admit further characterization of sets of numbers.

The natural numbers are not a group. For the addition they are lacking the neutral elements, and for adding and multiplications they are lacking inverse elements.

In contrast the group  $(\mathbb{Z}, +)$  is a commutative group with infinitely many elements.

**Example 2.9: The group  $(\mathbb{Z}, +)$** 

The numbers  $\mathbb{Z}$  with operation  $+$  form a group. This is demonstrated here by checking the group axioms.

a) addition of any two numbers provides a number:

$$\forall x, y \in \mathbb{Z} : (x + y) \in \mathbb{Z}.$$

b) The neutral element of the addition is 0:

$$\exists 0 \in \mathbb{Z} \forall z \in \mathbb{Z} : z + 0 = z = 0 + z.$$

c) For every element  $z \in \mathbb{Z}$  there is an inverse  $(-z) \in \mathbb{Z}$ :

$$\forall z \in \mathbb{Z} \exists (-z) \in \mathbb{Z} : z + (-z) = 0 = (-z) + z.$$

d) The addition of numbers is associative:

$$\forall z_1, z_2, z_3 \in \mathbb{Z} : z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

However, the numbers  $\mathbb{Z}$  still lack inverse elements of the multiplication. The rational numbers  $\mathbb{Q}$ , the real numbers  $\mathbb{R}$ , and the complex numbers  $\mathbb{C}$  are commutative groups for addition and multiplication (with the special rule that multiplication with 0 has no inverse element), and their elements also obey distributivity. Such sets are called number fields.

**Definition 2.7: Field**

A set  $(\mathbb{F}, +, \cdot)$  is called a *field* with neutral elements 0 and 1 for addition  $+$  and multiplication  $\cdot$ , respectively, when its elements comply with the following rules

- a)  $(\mathbb{F}, +)$  is a commutative group,
- b)  $(\mathbb{F} \setminus \{0\}, \cdot)$  is a commutative group,
- c) Addition and Multiplication are distributive:

$$\forall a, b, c \in \mathbb{F} : a \cdot (b + c) = a \cdot b + a \cdot c$$

*Remark 2.4.* For the multiplication of field elements one commonly suppresses the  $\cdot$  for the multiplication, writing e.g.  $ab$  rather than  $a \cdot b$ .

## 2.4 Vector Spaces

With the notions introduced in the preceding sections we can give now the formal definition of a vector space

**Definition 2.8: VectorSpace**

A *vector space*  $(V, \mathbb{F}, \oplus, \odot)$  is a set of *vectors*  $\vec{v}$  over a field  $(\mathbb{F}, +, \cdot)$  with binary operations  $\oplus : V \times V \rightarrow V$  and  $\odot : \mathbb{F} \times V \rightarrow V$  complying with the following rules

- a)  $(V, \oplus)$  is a commutative group
- b) associativity:  $\forall a, b \in \mathbb{F} \forall \vec{v} \in V : a \odot (b \odot \vec{v}) = (a \cdot b) \odot \vec{v}$
- c) distributivity 1:

$$\forall a, b \in \mathbb{F} \forall \vec{v} \in V : (a + b) \odot \vec{v} = (a \odot \vec{v}) \oplus (b \odot \vec{v})$$

- d) distributivity 2:

$$\forall a \in \mathbb{F} \forall \vec{v}, \vec{w} \in V : a \odot (\vec{v} \oplus \vec{w}) = (a \odot \vec{v}) \oplus (a \odot \vec{w})$$

*Remark 2.5.* It is common use  $+$  and  $\cdot$  instead of  $\oplus$  and  $\odot$ , respectively, with the understanding that it is clear from the context in the equation whether the symbols refer to operations involving vectors or only numbers.

Moreover, similar to the agreement for the multiplication of numbers, one commonly drop the  $\odot$  for the multiplication, writing e.g.  $a\vec{v}$  rather than  $a \odot \vec{v}$ .



**Example 2.10: Vector spaces: displacements in the plane**

For displacements we define the operation  $\oplus$  as concatenation of displacements, and  $\odot$  as increasing the length of the displacement by a given factor without touching the direction.

add proof

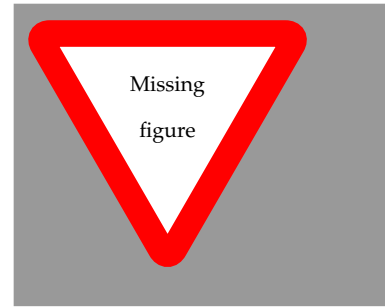


Figure 2.5: Graphical illustrations for Example 2.10

**Example 2.11: Vector spaces:  $\mathbb{R}^D$** 

For any  $D \in \mathbb{N}$  the  $D$ -fold cartesian product  $\mathbb{R}^D$  of the real numbers is a vector space over  $\mathbb{R}$  when defining the operation  $+$  and  $\cdot$  as

$$\forall \vec{a}, \vec{b} \in \mathbb{R}^D : \vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_D \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_D \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \dots \\ a_D + b_D \end{pmatrix}$$

$$\forall s \in \mathbb{R} \forall \vec{a} \in \mathbb{R}^D : s \cdot \vec{a} = s \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_D \end{pmatrix} = \begin{pmatrix} s a_1 \\ s a_2 \\ \dots \\ s a_D \end{pmatrix}$$

The checking of the properties of a vector space is suggested as an exercise for the reader.

**Example 2.12: Vector spaces: Polynomials of degree 2**

For a field  $\mathbb{F}$  the polynomials  $P$  of degree two in the variable  $x$  are defined as

$$P = \{\vec{p} = (p_0 + p_1 x + p_2 x^2) : p_0, p_1, p_2 \in \mathbb{F}\}$$

This set is a vector space with respect to the summation

$$\begin{aligned}\vec{p} + \vec{q} &= (p_0 + p_1 x + p_2 x^2) + (q_0 + q_1 x + q_2 x^2) \\ &= ((p_0 + q_0) + (p_1 + q_1) x + (p_2 + q_2) x^2)\end{aligned}$$

and the multiplication with a scalar  $s \in \mathbb{F}$

$$s \cdot \vec{p} = s \cdot (p_0 + p_1 x + p_2 x^2) = ((s p_0) + (s p_1) x + (s p_2) x^2)$$

add proof of vector-space properties

2.5 *Forces*

add "Forces"

2.6 *Scalar Products and Coordinates*

add "Scalar Products and Coordinates"

2.7 *Torques and cross products*

add "Torques and cross products"


2.8 *The mobile — a worked example*

add worked example "mobile"

2.9 *Problems**Rehearsing Concepts*

*Problem 2.1.* Checking group axioms

Which of the following sets are groups?

- a)  $(\mathbb{N}, +)$                       c)  $(\mathbb{Z}, \cdot)$                       e)  $(\{0\}, +)$   
 b)  $(\mathbb{Z}, +)$                       d)  $(\{+1, -1\}, \cdot)$                         $(\{1, \dots, 12\}, \oplus)$

where  $\oplus$  in f) reverts to adding as we do it on a clock,

e.g.  $10 \oplus 4 = 2$ .

**Problem 2.2. Euler's equation and trigonometric relations**

Euler's equation  $e^{ix} = \cos x + i \sin x$  relates complex values exponential functions and trigonometric functions.

- a) Sketch the position of  $e^{ix}$  in the complex plain, and indicate how Euler's equation is related to the Theorem of Pythagoras.  
 b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular one has  $e^{i(x+y)} = e^{ix} e^{iy}$ . Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about  $\sin(2x)$  and  $\cos(2x)$ ?

**Problem 2.3. Cartesian Coordinates**

- a) Mark the following points in a cartesian coordinate system:

$$(0;0) \quad (0;3) \quad (2;5) \quad (4;3) \quad (4;0)$$

Add the points  $(0;0)$   $(4;3)$   $(0;3)$   $(4;0)$ , and connect the points in the given order. What do you see?

- b) What do you find when drawing a line segment connecting the following points?

$$(0;0) \quad (1;4) \quad (2;0) \quad (-1;3) \quad (3;3) \quad (0;0)$$

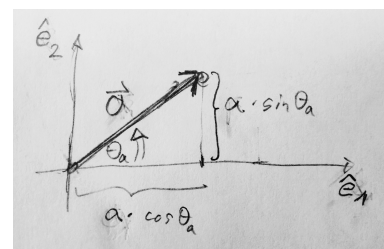
**Problem 2.4. Geometric and algebraic form of the scalar product**

The sketch in the margin shows a vector  $\vec{a}$  in the plane, and its representation as a linear combination of two orthonormal vectors  $(\hat{e}_1, \hat{e}_2)$ ,

$$\vec{a} = a \cos \theta_a \hat{e}_1 + a \sin \theta_a \hat{e}_2$$

Here,  $a$  is the length of the vector  $\vec{a}$ ,

and  $\theta_1 = \angle(\hat{e}_1, \vec{a})$ .



- a) Analogously to  $\vec{a}$  we will consider another vector  $\vec{b}$  with a representation

$$\vec{b} = b \cos \theta_b \hat{e}_1 + b \sin \theta_b \hat{e}_2$$

Employ the rules of scalar products, vector addition and multiplication with scalars to show that

$$\vec{a} \cdot \vec{b} = a b \cos(\theta_a - \theta_b)$$

**Hint:** Work backwards, expressing  $\cos(\theta_a - \theta_b)$  in terms of  $\cos \theta_a$ ,  $\cos \theta_b$ ,  $\sin \theta_a$ , and  $\sin \theta_b$ .

- b) As a shortcut to the explicit calculation of a) one can introduce the coordinates  $a_1 = a \cos \theta_a$  and  $a_2 = a \sin \theta_a$ , and write  $\vec{a}$  as a tuple of two numbers. Proceeding analogously for  $\vec{b}$  one obtains

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

How will the product  $\vec{a} \cdot \vec{b}$  look like in terms of these coordinates?

- c) How do the arguments in a) and b) change for  $D$  dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors  $\hat{e}_1, \dots, \hat{e}_D$ ?



What changes when the basis is not orthonormal?

What if it is not even orthogonal?

#### Problem 2.5. Angles between three balanced forces

Consider three forces  $\vec{F}_1$ ,  $\vec{F}_2$  and  $\vec{F}_p$  like in the rubber band example of the lecture, where I pull the band with force  $\vec{F}_p$  and this force is balanced by the forces due to the tension of the rubber band.

- a) Make a sketch of the setup where you indicate the angles  $\angle(\vec{F}_p, \vec{F}_1)$  as  $\theta_{1p}$  and  $\angle(\vec{F}_p, \vec{F}_2)$  as  $\theta_{2p}$ , respectively.
- b) Determine the condition for a balance of forces in the directions parallel to  $\vec{F}_p$  and parallel to  $\vec{F}_1$ .
- c) The result of (b) can be expressed as a conditions on  $F_p = |\vec{F}_p|$  as function of  $F_1$ ,  $F_2$ ,  $\theta_{1p}$  and  $\theta_{2p}$ , and on  $F_1$  as function of  $F_p$ ,  $F_2$ ,  $\theta_{1p}$  and  $\theta_{2p}$ . Insert the former condition into the latter one in order to eliminate  $F_p$ .

Hence, you find that  $F_1$  will be proportional to  $F_2$ , when the angles  $\theta_{1p}$  and  $\theta_{2p}$  are fixed. What does this reflect from a physical point of view?

**Hint:** What happens to the force balance when you fix the angle and increase  $\vec{F}_p$  by a factor  $\varphi$ .

- d) Employ trigonometric relations to show that the proportionality constant can be written as a ratio of two sines, i.e. one has

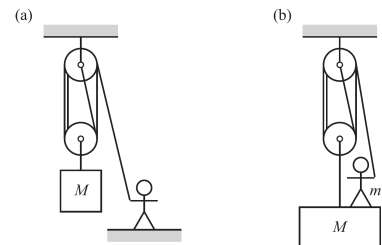
$$F_1 = \frac{\sin \alpha}{\sin \beta} F_2$$

How are the angles  $\alpha$  and  $\beta$  related to  $\theta_{1p}$  and  $\theta_{2p}$ ?

- e) Can you find a simpler way to derive the expression found in (d)?

**Problem 2.6. Tackling tackles and pulling pulleys**

- a) Which forces are required to hold the balance in sketch (a) and (b)?
- b) Let the sketched person and the weight have masses of 75 kg and 300 kg, respectively. Which power is required when to haul the line at a speed of 1 m/s.  
Hint: The power is defined here as the change of (a)  $Mgz(t)$  and (b)  $(M + m)gz(t)$ , per unit time, respectively.



*Practicing Concepts*

**Problem 2.7. Properties of right-angled triangles**


- a) Fill in the gaps for the values of the angle  $\theta$  in radians, and employ the symmetry of the trigonometric sine and cosine functions



to determine the values in the right columns

$\theta$		$\sin \theta$	$\cos \theta$
$^\circ$	rad		
0	_____	0	_____
30	_____	$\frac{1}{2}$	_____
45	_____	$\frac{\sqrt{2}}{2}$	_____
60	_____	$\frac{\sqrt{3}}{2}$	_____
90	_____	1	_____
120	_____	_____	_____
135	_____	_____	_____
150	_____	_____	_____
180	_____	_____	_____

b) Consider a right triangle where one of the angles is  $\theta$ . How are the length of its sides related to  $\sin \theta$  and  $\cos \theta$ ? Check that the Theorem of Pythagoras holds! Do you see a systematic for the angles?

c) Use the symmetries of the trigonometric functions to determine the values provided for  $\theta = \pi/4$ .

 Use the symmetries of the trigonometric functions and the trigonometric relation for  $\sin(2\theta)$  to determine the values provided for  $\theta = \pi/6$  and  $\theta = \pi/3$ .

  The values for  $\pi/10$ ,  $\pi/8$ , and  $\pi/5$  can also be stated explicitly in elementary form. Determine the expressions for these values!

**Problem 2.8. Linear Dependence of three vectors in 2D**

In the lecture I pointed out that every vector  $\vec{v} = (v_1, v_2)$  of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors  $\vec{a}$  and  $\vec{b}$ ,

$$\vec{v} = \alpha \vec{a} + \beta \vec{b}$$

In this exercise we revisit this statement for  $\mathbb{R}^2$  with the standard forms of vector addition and multiplication by scalars.

- a) Provide a triple of vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{v}$  such that  $\vec{v}$  can *not* be represented as a scalar combination of  $\vec{a}$  and  $\vec{b}$ .
- b) To be specific we will henceforth fix

$$\vec{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers  $\alpha$  and  $\beta$  such that

$$\vec{v} = \alpha \vec{a} + \beta \vec{b}$$

- c) Consider now also a third vector

$$\vec{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for  $(\alpha, \beta, \gamma)$  such that

$$\vec{v} = \alpha \vec{a} + \beta \vec{b} + \gamma \vec{c}$$

What is the general constraints on  $(\alpha, \beta, \gamma)$  such that  $\vec{v} = \alpha \vec{a} + \beta \vec{b} + \gamma \vec{c}$ .

What does this imply on the number of solutions?

- d) Discuss now the linear dependence of the vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  by exploring the solutions of

$$\vec{0} = \alpha \vec{a} + \beta \vec{b} + \gamma \vec{c}$$

How are the constraints for the null vector related to those obtained in part c)?

**Problem 2.9. Torques acting on a ladder**

The sketch in the margin shows the setup of a ladder leaning to the roof of a hut. The indicated angle from the downwards vertical

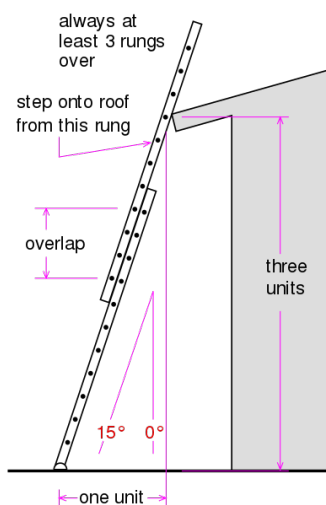


Figure 2.6: Sketch is due to Bradley and the vector image to Sarang [Public domain from wikimedia]

to the ladder will be denoted as  $\theta$ . There is a gravitational force of magnitude  $Mg$  acting on a ladder of mass  $M$ . At the point where it leans to the roof there is a normal force of magnitude  $F_r$  acting from the roof to the ladder. At the ladder feet there is a normal force to the ground of magnitude  $F_g$ , and a tangential friction force of magnitude  $\gamma F_f$ . This is again the sketch to the ladder leaning to the roof of a hut. The angle from the downwards vertical to the ladder is denoted  $\theta$ . There is a gravitational force of magnitude  $Mg$  acting on a ladder. At the point where it leans to the roof there is a normal force of magnitude  $F_r$ . At the ladder feet there is a normal force to the ground of magnitude  $F_g$ , and a tangential friction force of magnitude  $F_f$ .

- a) In principle there also is a friction force  $\gamma_r F_r$  acting at the contact from the ladder to the roof. Why is it admissible to neglect this force?

Remark: There are at least two good arguments.

- b) Determine the vertical and horizontal force balance for the ladder. Is there a unique solution?

- c) The feet of the ladder start sliding when  $F_f$  exceeds the maximum static friction force  $\gamma F_g$ . What does this condition entail for the angle  $\theta$ ?

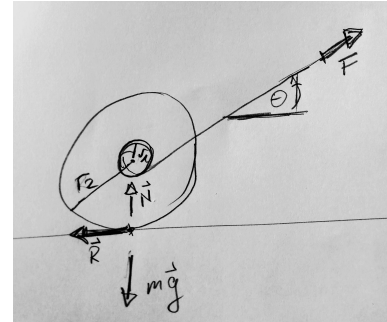
Assume that  $\gamma \simeq 0.3$ . What does this imply for the critical angle  $\theta_c$ .

- d) Where does the mass of the ladder enter the discussion? Do you see why?
- e) Determine the torque acting on the ladder. Does it matter whether you consider the torque with respect to the contact point to the roof, the center of mass, or the foot of the ladder?
- f) The ladder will slide when the modulus of the friction force  $F_f$  exceeds a maximum value  $\mu_S F_g$  where  $\mu$  is the static friction coefficient for of the ladder feet on the ground. For metal feet on a wooden ground it takes a value of  $\mu_S \simeq 2$ . What does that tell about the angles when the ladder starts to slide?
- g) Why does a ladder commonly starts sliding when when a man has climbed to the top? Is there anything one can do against it? Is that even true, or just an urban legend?



**Problem 2.10. Walking a yoyo**

The sketch to the right shows a yoyo of mass  $m$  standing on the ground. It is held at a chord that extends to the top right. There are four forces acting on the yoyo: gravity  $m\vec{g}$ , a normal force  $\vec{N}$  from the ground, a friction force  $\vec{R}$  at the contact to the ground, and the force  $\vec{F}$  due to the chord. The chord is wrapped around an axle of radius  $r_1$ . The outer radius of the yoyo is  $r_2$ .



- Which conditions must hold such that there is no net force acting on the center of mass of the yoyo?
- For which angle  $\theta$  will the torque vanish?
- Perform an experiment: What happens for larger and for smaller angles  $\theta$ ? How does the yoyo respond when fix the height where you keep the chord and pull continuously?

**Proofs****Problem 2.11. Different basis for polynomials**

We consider the set of polynomials  $\mathbb{P}_N$  of degree  $N$  with real coefficients  $p_n, n \in \{0, \dots, N\}$ ,

$$\mathbb{P}_N := \left\{ \vec{p} = \left( \sum_{k=0}^N p_n x^k \right) \text{ mit } p_n \in \mathbb{R}, n \in \{0, \dots, N\} \right\}$$

- Demonstrate that  $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$  is a vector space when one adopts the operations

$$\forall \vec{p} = \left( \sum_{k=0}^N p_n x^k \right) \in \mathbb{P}_N, \vec{q} = \left( \sum_{k=0}^N q_n x^k \right) \in \mathbb{P}_N, \text{ and } c \in \mathbb{R} :$$

$$\vec{p} + \vec{q} = \left( \sum_{k=0}^N (p_k + q_k) x^k \right) \quad \text{and} \quad c \cdot \vec{p} = \left( \sum_{k=0}^N (c p_k) x^k \right).$$

- Demonstrate that

$$\vec{p} \cdot \vec{q} = \left( \int_0^1 dx \left( \sum_{k=0}^N p_k x^k \right) \left( \sum_{j=0}^N q_j x^j \right) \right),$$

establishes a scalar product on this vector space.

- Demonstrate that the three polynomials  $\vec{b}_0 = (1)$ ,  $\vec{b}_1 = (x)$  und  $\vec{b}_2 = (x^2)$  form a basis of the vector space  $\mathbb{P}_2$ : For each

polynomial  $\vec{p}$  aus  $\mathbb{P}_2$  there are real numbers  $x_k, k \in \{0, 1, 2\}$ , such that  $\vec{p} = x_0 \vec{b}_0 + x_1 \vec{b}_1 + x_2 \vec{b}_2$ . However, in general we have  $x_i \neq \vec{p} \cdot \vec{b}_i$ . Why is that?

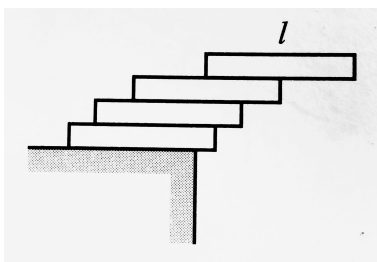
Hint: Is this an orthonormal basis?

- (d) Demonstrate that the three vectors  $\hat{e}_0 = (1)$ ,  $\hat{e}_1 = \sqrt{3}(2x - 1)$  and  $\hat{e}_2 = \sqrt{5}(6x^2 - 6x + 1)$  are orthonormal.
- (e) Demonstrate that every vector  $\vec{p} \in \mathbb{P}_2$  can be written as a scalar combination of  $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$ ,

$$\vec{p} = (\vec{p} \cdot \hat{e}_0) \hat{e}_0 + (\vec{p} \cdot \hat{e}_1) \hat{e}_1 + (\vec{p} \cdot \hat{e}_2) \hat{e}_2.$$

Hence,  $(\hat{e}_0, \hat{e}_1, \hat{e}_2)$  form an orthonormal basis of  $\mathbb{P}_2$ .

- \*(f) Find a constant  $c$  and a vector  $\hat{n}_1$ , such that  $\hat{n}_0 = (cx)$  and  $\hat{n}_1$  form an orthonormal basis of  $\mathbb{P}_1$ .



**Problem 2.12. Piling bricks**

Christmas is approaching, and Germans consume enormous amounts of chocolate. If you happen to come across a considerable pile of chocolate bars (or beer mats, or books, or anything else of that form) I recommend the following experiment:

- a) We consider  $N$  bars of length  $l$  piled on a table. What is the maximum amount that the topmost bar can reach beyond the edge of the table.
- b) The sketch above shows the special case  $N = 4$ .  
However, what about the limit  $N \rightarrow \infty$ ?

*Transfer and Bonus Problems, Riddles*

**Problem 2.13. Systems of linear equations**

A system of  $N$  linear equations of  $M$  variables  $x_1, \dots, x_M$  comprises  $N$  equations of the form

$$\begin{aligned} b_1 &= a_{11} x_1 + a_{12} x_2 + \cdots + a_{1M} x_M \\ b_2 &= a_{21} x_1 + a_{22} x_2 + \cdots + a_{2M} x_M \\ &\vdots \\ b_N &= a_{N1} x_1 + a_{N2} x_2 + \cdots + a_{NM} x_M \end{aligned}$$

where  $b_i, a_{ij} \in \mathbb{R}$  for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ .

- a) Demonstrate that the linear equations  $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$  form a vector space when one adopts the operations

$$\forall \vec{p} = [p_0 = p_1 x_1 + p_2 x_2 + \cdots + p_M x_M] \in \mathbb{L}_N,$$

$$\vec{q} = [q_0 = q_1 x_1 + q_2 x_2 + \cdots + q_M x_M] \in \mathbb{L}_N,$$

$$c \in \mathbb{R} :$$

$$\vec{p} + \vec{q} = [p_0 + q_0 = (p_1 + q_1) x_1 + (p_2 + q_2) x_2 + \cdots + (p_M + q_M) x_M]$$

$$c \cdot \vec{p} = [c p_0 = c p_1 x_1 + c p_2 x_2 + \cdots + c p_M x_M].$$

How do these operations relate to the operations performed in Gauss elimination to solve the system of linear equations?

- b) The system of linear equations can also be stated in the following form

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M$$

$$\Leftrightarrow \vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_M \vec{a}_M$$

where  $\vec{b}$  is expressed as a linear combination of  $\vec{a}_1, \dots, \vec{a}_M$  by means of the numbers  $x_1, \dots, x_M$ . What do the conditions on linear independence and representation of vectors by means of a basis tell about the existence and uniqueness of the solutions of a system of linear equations.

**Problem 2.14. Eagle and Hedgehog**

- a) You are looking from South into a valley and see a hedgehog that sits 10 m to the right of a tree on in the grass. From far right above an eagle is attacking. You monitor its position based on coordinates to the right and vertical from the foot of the tree, indicating distances in meter:

$$(10.9; 28) \quad (11.5; 44) \quad (12.1; 62) \quad (13.2; 99)$$

Will the eagle catch the hedgehog?

- b) A friend is taking a movie of the same incidence, observing it from the East. From the movie you extract the position of the eagle to the North and vertical with respect of the foot of the

tree.

(1.9; 27) (2.4; 44) (3.1; 62) (3.8; 85)

Will the eagle catch the hedgehog?

- c) A forester, who knows the animals really well, is telling you that the hedgehog will typically start moving North with a speed of about  $1 \text{ m s}^{-1}$  when he notices the eagle coming down. Will the eagle catch the hedgehog when it is coming down with a speed of  $50 \text{ m s}^{-1}$  and when the hedgehog notices the eagle at a height of 50 m?

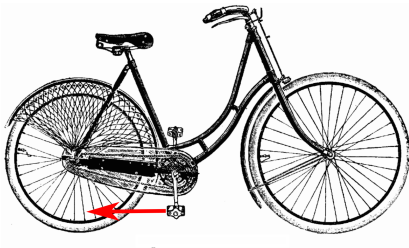


Figure 2.7: The picture of the bicycle is due to Otto Lueger, Damen-fahrrad 1904 [Public domain, wikimedia].

**Problem 2.15. Where will the bike go?**

Consider the picture of the bicycle to the left. The red arrow indicates a force that is acting on the paddle in backward direction.

Will the bicycle move forwards or backwards?

Take a bike and do the experiment!

# 3

## *Newton's Laws*

In Chapter 2 we explored how several forces that act on a body can be subsumed into a net total force and torque. The body stays in rest, say at position  $\vec{q}_0$ , when the net force and torque vanish. Now we explore how the forces induce motion and how the position of the body evolves in time,  $\vec{q}(t)$ , when it is prepared with an initial condition  $\vec{q}(t_0) = \vec{q}_0$  at the initial time  $t_0$ .

At the end of this chapter we will be able to discuss the likelihood for injuries in different types of accidents, be it men or cat or mice. Why do the cats go away unharmed in most cases when they fall from a balcony, while an old professor should definitely avoid such a fall.

### *3.1 Motivation and Outline: What is causing motion?*

Every now and then I make the experience that I sit in a train, reading a book. Then I look out of the window, realize that we are passing a train, feeling happy that we are further approaching my final destination; and then I realize that the train is moving and my train is still in the station. Indeed, the motion of objects in my compartment is exactly identical, no matter whether it is at rest or moves with a constant velocity; be it zero in the station, at 15 m/s in a local commuter train, or 75 m/s in a Japanese high-speed train. However, changes of speed matter. I forcefully experience the force exerted on the train during an emergency break.

Modern physics was born when Galilei and Newton formalized this experience by saying that bodies (e.g. the set of bodies in the compartment of a train) move in a straight line with a constant velocity as long as there is no net force acting on the bodies. A pulley pushed by a mine worker is moving at constant speed because the

force induced by the mine worker is exactly balanced by friction forces. All that physics has to say about the cause of motion is that it is due to some force that has been working in the past. Its speed and direction changes if and only if a force is acting.

### Outline

In the first part of this chapter we will relate temporal changes of positions and velocities to time derivatives. Subsequently, we can formulate equations of motion that relate these changes to forces. To this end we will have to introduce the mathematical concept of differential equations. The last part of the chapter deals with strategies to find solutions and characterize sets of solutions of differential equations.

mass	$m$
position	$\vec{q}(t)$
velocity	$\dot{\vec{q}}(t), \vec{v}(t)$
acceleration	$\ddot{\vec{q}}(t)$
forces	$\vec{F}_\alpha(\vec{q}, t)$

Table 3.1: Notations adopted to describe the motion of a particle. A single dot denotes the time derivative, and double dot the second derivative with respect to time.

### 3.2 Time derivatives of vectors

In this section we consider the motion of a particle with mass  $m$  that is at position  $\vec{q}(t)$  at time  $t$ . When it moves, then its average velocity  $\vec{v}_{\text{av}}(t, \Delta t)$  during the time interval  $[t, t + \Delta t]$  is

$$\vec{v}_{\text{av}}(t, \Delta t) = \frac{\vec{q}(t + \Delta t) - \vec{q}(t)}{\Delta t}$$

When the limit  $\lim_{\Delta t \rightarrow 0} \vec{v}_{\text{av}}(t, \Delta t)$  exists<sup>1</sup> we can define the velocity of the particle at time  $t$ ,

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{q}(t + \Delta t) - \vec{q}(t)}{\Delta t} \quad (3.2.1)$$

The velocity is then the time derivative of the position, and in an immediate generalization of the time derivative of scalar functions we also write

$$\dot{\vec{q}}(t) = \vec{v}(t) = \frac{d\vec{q}(t)}{dt}$$

Finally, we point out that the components of the time derivative of a vector amount to the derivatives of the components.

<sup>1</sup> The discussion of this limit of general functions is a core topic of vector calculus. For our present purpose the intuitive understanding based on the idea that  $\vec{q}(t + \Delta t) \simeq \vec{q}(t) + \Delta t \vec{v}(t)$  provides the right idea. To provide a hint for the origin of the mathematical subtleties we point out that the approximation works unless there is an *instantaneous* collision with a wall at some point in the time interval  $]t, t + \Delta t[$ . In physics we try our luck, and fix the problem when we face it. Indeed, upon a close look there are no instantaneous collisions in physics.

**Theorem 3.1: Time derivatives of vectors**

Let  $\vec{a}(t)$  be a vector with time-dependent components  $a_i(t)$  with respect to orthonormal basis  $\{\hat{e}_i, i = 1 \dots D\}$  that is fixed in time.

Then  $\dot{\vec{a}}(t) = \sum_i \dot{a}_i(t) \hat{e}_i$ , i.e. the components of  $\dot{\vec{a}}(t)$  amount to the time derivatives of the components of  $\vec{a}(t)$ .

*Proof.* For each time we have  $\vec{a}(t) = \sum_i a_i(t) \hat{e}_i$  where it is understood that the sum runs over  $i = 1 \dots D$ . We insert this into the definition, Equation (3.2.1), of the time derivative and use the linearity of scalar products with vectors to obtain

$$\begin{aligned} \dot{\vec{a}}(t) &= \lim_{\Delta t \rightarrow 0} \frac{\vec{a}(t + \Delta t) - \vec{a}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\sum_i a_i(t + \Delta t) \hat{e}_i - \sum_i a_i(t) \hat{e}_i}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \sum_i \hat{e}_i \frac{a_i(t + \Delta t) - a_i(t)}{\Delta t} = \sum_i \hat{e}_i \lim_{\Delta t \rightarrow 0} \frac{a_i(t + \Delta t) - a_i(t)}{\Delta t} \\ &= \sum_i \hat{e}_i \dot{a}_i(t) \end{aligned}$$

The subtle step here, from a mathematical point of view, is the swapping of the limit and the sum in the second line of the argument. Courses on vector calculus will spell out the assumptions needed to justify this step (or, more interestingly from a physics perspective, under which conditions it fails).  $\square$

The change of the velocity will be denoted as acceleration. Based on an analogous argument as for the velocity, it will be written as a time derivative

**Definition 3.1: Acceleration**

The time derivative of the velocity  $\vec{v}(t) = \dot{\vec{q}}(t)$  will be denoted as acceleration, and written as

$$\frac{d\vec{v}(t)}{dt} = \dot{\vec{v}}(t) = \ddot{\vec{q}}(t)$$

In the next section it will be related to the action of forces  $\vec{F}(\vec{q}, t)$  acting on a particle that resides at the position  $\vec{q}$  at time  $t$ .

## 3.3 Newton's Axioms

In Section 4.1 we referred to a train compartment to point out that physical observations will be the same — irrespective of the velocity of its motion, as long as it is constant. As setting where we

perform an experiment is denoted as reference frame, and reference frames that move with constant velocity are called inertial systems.

### Definition 3.2: Reference Frames and Inertial Systems

A reference frame  $(\vec{Q}, \{\hat{e}_i(t), i = 1 \cdots D\})$  is an agreement about the, in general time dependent, position of the origin  $\vec{Q}(t)$  of the coordinate system and a set of orthonormal basis vectors  $\{\hat{e}_i(t), i = 1 \cdots D\}$ , that are adopted to indicate the positions of particles in a physical model.

The reference frame refers to an *inertial system* when it does not rotate and when it moves with a constant velocity, i.e., if and only if  $\ddot{\vec{Q}} = \vec{0}$  and  $\dot{\hat{e}}_i = \vec{0}$  for all  $i \in \{1 \cdots D\}$ .

### 1st Law

As long as a reference frame moves with a constant velocity, it feels like at rest. Physical measurements can only detect acceleration.

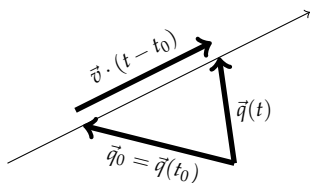
This is expressed by

### Axiom 3.1: Newton's 1st law

The velocity of a particle moving in an inertial system is constant, unless a (net) force is acting on the particle,

$$\begin{aligned} \forall t \geq t_0 : \vec{F}(t) = \vec{0} &\Leftrightarrow \dot{\vec{q}}(t) = \vec{v} = \text{const} \\ &\Leftrightarrow \vec{q}(t) = \vec{q}_0 + \vec{v}(t - t_0) \end{aligned}$$

as sketched in the margin.



The particle moves then in a straight line with a constant speed. Indeed, when a particle moves with the constant velocity  $\vec{v} = \dot{\vec{q}}(t)$  in the reference frame  $(\vec{Q}, \{\hat{e}_i(t), i = 1 \cdots D\})$  then it is at rest in the alternative reference frame  $(\vec{Q} + \vec{v}t, \{\hat{e}_i(t), i = 1 \cdots D\})$ . Therefore, in the latter coordinate system the particle is at rest, and it will remain at rest when it is not perturbed by a net external force.

*Proof.* We choose the origin for the frame of reference to describe the position of the particle to be

$$\vec{Q} = \vec{q}(t)$$

By construction the particle is always in the origin of this coordinate frame, i.e., at rest, and for  $\vec{q}(t) = \vec{q}_0 + \vec{v}(t - t_0)$  the coordinate



frame is an inertial frame because

$$\dot{\vec{Q}} = \dot{\vec{q}}(t) = \frac{d}{dt}(\vec{q}_0 - \vec{v} t_0) + \vec{v} t + \vec{v} = \vec{v} = \text{const}$$

where the first two terms vanish because  $\vec{q}_0$ ,  $\vec{v}$ , and  $t_0$  are constant. □

### 2nd Law

Newton's second law spells out how the velocity of the particle changes when there is a force.

#### Axiom 3.2: Newton's 2nd law

The change,  $\ddot{\vec{q}}(t)$ , of the velocity of a particle,  $\dot{\vec{q}}(t)$ , at position,  $\vec{q}(t)$ , is proportional to the sum of acting forces  $\vec{F}_\alpha$  with a proportionality factor  $m$ ,

$$\ddot{\vec{q}}(t) = m \sum_{\alpha} \vec{F}_{\alpha}(t)$$

*Remark 3.1.* In general the time dependence of the forces can be decomposed into three contributions

- a) An implicit time dependence,  $\vec{F}(\vec{q}(t))$ , when the force depends on the position,  $\vec{q}(t)$  of the particle. For instance, for a Hookian spring with spring constant  $k$  one has,  $\vec{F}(\vec{q}) = -k \vec{q}$
- b) An implicit time dependence,  $\vec{F}(\dot{\vec{q}}(t))$ , when the force depends on the velocity,  $\dot{\vec{q}}(t)$  of the particle. For instance, the sliding friction for a particle with mass  $m$  and friction coefficient  $\gamma$  is,  $\vec{F}(\dot{\vec{q}}) = -m \gamma \dot{\vec{q}}$
- c) An explicit time dependence when the force is changing in time. For instance, when pushing a child sitting on a swing one will only push when the swing is moving in forward direction.

Typically, one explicitly sorts out these dependences and writes

$$\ddot{\vec{q}}(t) = m \sum_{\alpha} \vec{F}_{\alpha}(\vec{q}(t), \dot{\vec{q}}(t), t)$$

**Example 3.1: Particle moving in the gravitational field**

The gravitational field induces a constant force  $m\vec{g}$  on a particle with mass  $m$ . Let it have velocity  $\vec{v}_0$  at time  $t_0$  when it is taking off from the position  $\vec{q}_0$ . Then Newton's 2nd law states that  $\ddot{\vec{q}}(t) = \vec{g}$ , and this equation must be solved subject to the initial conditions  $\vec{q}(t_0) = \vec{q}_0$  and  $\dot{\vec{q}}(t_0) = \vec{v}$ . By working out the derivatives one readily checks that this is given for

$$\vec{q}(t) = \vec{q}_0 + \vec{v}(t - t_0) + \frac{1}{2}\vec{g}(t - t_0)^2$$

**Example 3.2: Particle moving in a circle**

Let a particle of mass  $m$  move with constant speed in a circle of radius  $R$  such that its position can be written as

$$\vec{q}(t) = \begin{pmatrix} R \cos(\omega t) \\ R \sin(\omega t) \end{pmatrix}$$

with a constant angular velocity  $\omega$ . Then its velocity and acceleration take the form

$$\begin{pmatrix} -\omega R \sin(\omega t) \\ \omega R \cos(\omega t) \end{pmatrix} \quad \text{and} \quad \ddot{\vec{q}}(t) = \begin{pmatrix} -\omega^2 R \cos(\omega t) \\ -\omega^2 R \sin(\omega t) \end{pmatrix} = -\omega^2 R \vec{q}(t)$$

The speed is constant, taking the value  $\sqrt{\dot{\vec{q}} \cdot \dot{\vec{q}}} = \omega R$  and the force is antiparallel to  $\vec{q}$  with magnitude  $m\omega^2 R$ . Moreover,  $\dot{\vec{q}} \cdot \vec{F} = 0$  at all times. In this case the force only changes the direction, and not the modulus of the velocity.

*3rd Law*

Newton's third law states that the reference frame does not matter for the description of the evolution of two particles, even when they interact with each other — i.e. when they exert forces on each other. Consider for instance the motion of two particles of the same mass  $m$  that reside at the positions  $\vec{q}_1(t)$  and  $\vec{q}_2(t)$ . We decide to observe them from a position right in the middle between the two particles  $\vec{Q} = (\vec{q}_1(t) + \vec{q}_2(t))/2$ . In accordance with observation, we require that this is an inertial frame of reference in the absence of external forces, such that  $\ddot{\vec{Q}} = \vec{0}$  according to Newton's first law. However,

Newton's second law implies that also

$$\vec{0} = 2m\ddot{Q} = m\ddot{q}_1 + m\ddot{q}_2 = \vec{F}_1 + \vec{F}_2$$

where  $\vec{F}_1 = m\ddot{q}_1$  and  $\vec{F}_2 = m\ddot{q}_2$  are the forces acting on particle 1 and 2, respectively. Up to a change of sign the forces are the same,  $\vec{F}_1 = -\vec{F}_2$ . This action-reaction principle is stipulated by

#### Axiom 3.3: Newton's 3rd law

Forces act in pairs:

*actio* when a body  $A$  is pushing a body  $B$  with force  $\vec{F}_{A \rightarrow B}$

*reactio* then the body  $B$  is pushing  $A$  with force  $\vec{F}_{B \rightarrow A} = -\vec{F}_{A \rightarrow B}$

The forces in such a pair are always balanced,  $\vec{F}_{A \rightarrow B} + \vec{F}_{B \rightarrow A} = \vec{0}$ .

#### Example 3.3: Fixing a hammok at a tree

When you lie in a hammok that is fixed at a tree, your hammok exerts a force  $\vec{F}_h$  on the tree (*actio*). The hammok stays where it is because the tree pulls back with exactly the same force  $-\vec{F}_h$ , up to a change of sign (*reactio*).

#### Example 3.4: Ice scaters

- When two ice scaters of the same mass push each other starting from a position at rest, then they will move in opposite directions with the same speed (unless they break).
- When they have masses  $m_1$  and  $m_2$  their velocities will be related by  $m_1 \vec{v}_1 + m_2 \vec{v}_2 = \vec{0}$  because  $\vec{v}_1 = \vec{v}_2 = \vec{0}$  initially, and  $m_1 \dot{\vec{v}}_1 + m_2 \dot{\vec{v}}_2 = \vec{F}_1 + \vec{F}_2 = \vec{0}$  at any instant of time.

#### Punchline

Newton's equations are stated nowadays in terms of derivatives, a concept in calculus that has been pioneered by Leibnitz.<sup>2</sup> In this language they take the following form for a particle of mass  $m$  that

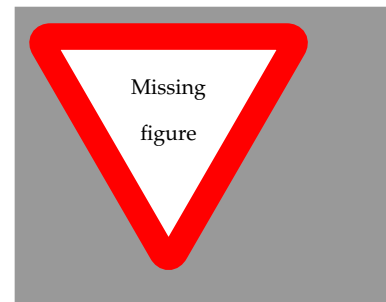


Figure 3.1: Graphical illustrations of forces involved in hanging a hammat on a tree.

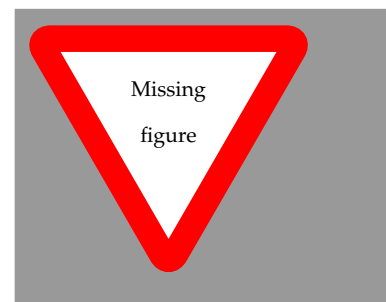


Figure 3.2: Graphical illustrations of motion of the two ice-scaters of Example 3.4.

<sup>2</sup> Even though the principles have been understood by Newton which led to a very long fight for authorship and fame.

is at position  $\vec{q}(t)$  at time  $t$ ,

$$\begin{aligned}\dot{\vec{q}}(t) &= \vec{v}(t) \\ \dot{\vec{v}}(t) &= \frac{1}{m} \vec{F}_{\text{tot}}(\vec{q}(t), \vec{v}(t), t)\end{aligned}$$

Prior to Newton physical theories adopted the aristotelian point of view that  $\vec{v}$  is proportional to the force. Indeed in those days many scientists were regularly inspecting mines, and from the perspective of pushing minecarts it is quite natural to assert that their velocity is proportional to the pushing force. Galilei's achievement is to add the 'tot' of the force side of the equation, pointing out that there also is a friction force acting on the minecart. Newton's achievement is to add the 'dot' on the left side of the equation, stating that the velocity stays constant when the pushing force and the friction force balance.

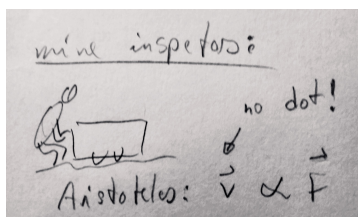


Figure 3.3: Illustration about the pre-newtonian understanding of the relation between force and the velocity of a body.

### Example 3.5: Pushing a minecart

The motion of the minecart is one-dimensional along its track such that the position,  $q$ , velocity,  $x$ , and forces are one-dimensional, i.e., scalar functions. Once the minecart is moving it experiences a friction force  $F_f = -\gamma v$ , that is growing with speed,  $v$ . Now, let the mineworker push with a constant force  $F_M$ . Consequently,

$$\ddot{q} = \dot{v} = F_{\text{tot}} = F_M - \gamma v$$

The minecart travels with constant velocity  $\dot{v} = 0$ , when the attacking forces balance, i.e., for  $v_c = F_M/\gamma$ . However, when starting from a different velocity,  $v(t_0) = v_0$ , we rather find an exponential approach to the asymptotic velocity,

$$v(t) = v_c + (v_0 - v_c) e^{-\gamma(t-t_0)}$$

After all,  $v(t_0) = v_c + (v_0 - v_c) = v_0$  and

$$\dot{v}(t) = (v_0 - v_c) (-\gamma) e^{-\gamma(t-t_0)} = -\gamma (v(t) - v_c) = -\gamma v(t) + F_M$$

The advantage of the Newtonian approach above earlier modeling attempts is that it makes a quantitative prediction about the asymptotic velocity, and that it also addresses the regime where the velocity is changing, e.g., when the minecart is taking up speed.

## 3.4 Constants of motion (CM)

In the previous section we saw that Newton's laws can be expressed as equations relating the second derivative of the position of a particle to the forces acting on the particle. The forces are determined as part of setting up the physical model. Subsequently, determining the time dependence of the position is a mathematical problem. Often it can be solved by finding constraints on the solution that must hold for all times. Such a constraint is called a

**Definition 3.3: Constant of motion**

A function  $\mathcal{C}(\vec{q}, \dot{\vec{q}}, t)$  is a *constant of motion (CM)* iff its time derivative vanishes,

$$\frac{d}{dt}\mathcal{C}(\vec{q}, \dot{\vec{q}}, t) = 0$$

It provides us with an opportunity to take a closer look at the expressions that emerge when taking derivatives of functions with arguments that are vectors. In order to evaluate the time derivative of  $\mathcal{C}$  we write  $\vec{q} = (q_1, \dots, q_D)$ , and apply the chain rule

$$\begin{aligned} \frac{d}{dt}\mathcal{C}(\vec{q}(t), \dot{\vec{q}}(t), t) &= \frac{d}{dt}\mathcal{C}(q_1(t) \cdots q_D(t), \dot{q}_1(t) \cdots \dot{q}_D(t), t) \\ &= \sum_{i=1}^D \frac{dq_i}{dt} \frac{\partial \mathcal{C}}{\partial q_i} \Big|_{\substack{q_j \text{ with } j \neq i \\ \dot{\vec{q}}, t}} + \sum_{i=1}^D \frac{d\dot{q}_i}{dt} \frac{\partial \mathcal{C}}{\partial \dot{q}_i} \Big|_{\substack{\dot{q}_j \text{ with } j \neq i \\ q, t}} + \frac{\partial \mathcal{C}}{\partial t} \Big|_{\vec{q}, \dot{\vec{q}}} \end{aligned}$$

In this expression the operator  $\partial$  is called 'partial', and the derivative  $\partial \mathcal{C} / \partial q_i \Big|_{\text{conditions}}$  is denoted as 'partial derivative of  $\mathcal{C}$  with respect to  $q_i$  for fixed ...' arguments that are explicitly indicated in the conditions. In other words, for the purpose of calculating the partial derivative, we consider  $\mathcal{C}$  to be a function of only the single argument  $q_i$  based on the conditions spelled out as subscript after the vertical bar  $|$ . For sake of a more compact notation we also write  $\partial_{q_i} \mathcal{C}$  rather than  $\partial \mathcal{C} / \partial q_i$ . Moreover, when it is clear from the context which conditions are adopted, they will typically not be specified. From a physicist's perspective this reduces clutter in the equations.

An more compact notation that is still more transparent is achieved by observing that the expressions in the sums amount to writing out in components a scalar product of  $\vec{q}$  and  $\dot{\vec{q}}$  with vectors that are obtained by the partial derivatives. These vectors are

denoted *gradients* with respect to  $\vec{q}$  and  $\dot{\vec{q}}$ , and they will be written as

$$\nabla_{\vec{q}}\mathcal{C} = \begin{pmatrix} \partial_{q_1}\mathcal{C} \\ \vdots \\ \partial_{q_D}\mathcal{C} \end{pmatrix} \quad \text{and} \quad \nabla_{\dot{\vec{q}}}\mathcal{C} = \begin{pmatrix} \partial_{\dot{q}_1}\mathcal{C} \\ \vdots \\ \partial_{\dot{q}_D}\mathcal{C} \end{pmatrix}$$

such that

$$\frac{d}{dt}\mathcal{C}(\vec{q}(t), \dot{\vec{q}}(t), t) = \dot{\vec{q}} \cdot \nabla_{\vec{q}}\mathcal{C} + \ddot{\vec{q}} \cdot \nabla_{\dot{\vec{q}}}\mathcal{C} + \frac{\partial\mathcal{C}}{\partial t}$$

In terms of the phase-space coordinates  $\Gamma = (\vec{q}, \dot{\vec{q}})$  one can even adopt the even more compact notation

$$\frac{d}{dt}\mathcal{C}(\vec{q}(t), \dot{\vec{q}}(t), t) = \dot{\vec{q}} \cdot \nabla_{\Gamma}\mathcal{C} + \frac{\partial\mathcal{C}}{\partial t}$$

We conclude the section by introducing some important physical quantities that are constants of the motion in specific settings.

### *The kinetic energy*

When no forces are acting on a particle,  $\vec{F}_{\text{tot}} = \vec{0}$ , it moves with constant velocity. All functions that depend only on the velocity will then be constant. In particular this holds for the kinetic energy,  $T$ , that will play a very important role in the following.

#### **Theorem 3.2: Conservation of kinetic energy**

The *kinetic energy*  $T = \frac{m}{2} \dot{\vec{q}}^2$  of a particle is conserved iff no net force acts on the particle, i.e., iff  $\vec{F}_{\text{tot}} = \vec{0}$ .

*Proof.*

$$\begin{aligned} \frac{d}{dt}T &= \frac{m}{2} \frac{d}{dt} \sum_i \dot{q}_i \cdot \dot{q}_i \\ &= m \sum_i \dot{\vec{q}} \cdot \ddot{\vec{q}} = \dot{\vec{q}} \cdot (m \ddot{\vec{q}}) = \dot{\vec{q}} \cdot \vec{F}_{\text{tot}} = 0 \end{aligned}$$

In the last two steps we used Newton's 2nd law, and the assumption that  $\vec{F}_{\text{tot}} = \vec{0}$ . □

### *Work and total energy*

From a physics perspective, work is performed when a body is moved that is experiencing an external force.

- When the force  $\vec{F}$  is constant along a path of displacement  $\vec{s} = \vec{q}_1 - \vec{q}_0$ , from a position  $\vec{q}_0$  to the position  $\vec{q}_1$ , then the work  $W$  amounts to the scalar product  $W = \vec{F} \cdot \vec{s}$ .
- When the force changes upon moving along the path, we parameterize the motion along the path by time,  $\vec{q}(t)$ , with  $\vec{q}(t_0) = \vec{q}_0$  and  $\vec{q}(t_1) = \vec{q}_1$  and break it into sufficiently small pieces  $\vec{s}_i = \dot{\vec{q}}(t_i) \Delta t$  where the force  $\vec{F}_i = \vec{F}(t_i)$  may be assumed to be constant. Then

$$W = \sum_i \vec{F}_i \cdot \vec{s}_i = \lim_{\Delta t \rightarrow 0} \vec{F}_i \cdot \dot{\vec{q}} \Delta t = \int_{t_0}^{t_1} \vec{F}(t) \cdot \dot{\vec{q}}(t) dt = \int_{\vec{q}(t)} \vec{F} \cdot d\vec{q}$$

The last equality should be understood here as a definition of the final expression that is interpreted here in the spirit of the substitution rule of integration.

#### Definition 3.4: Work and Line Integrals

The *work*,  $W$ , of a particle that performs a path  $\vec{q}$  under the influence of a force  $\vec{F}(t)$  amounts to the result of the *line integral*

$$W = \int_{\vec{q}} \vec{F} \cdot d\vec{q}$$

When the path is parameterized by time, then  $W$  amounts to the time integral of dissipated power  $P(t) = \vec{F}(t) \cdot \dot{\vec{q}}(t)$ ,

$$W = \int \vec{F}(t) \cdot \dot{\vec{q}}(t) dt = \int P(t) dt$$

*Remark 3.2.* a) The scalar product  $\vec{F} \cdot d\vec{q}$  or  $P(t) = \vec{F}(t) \cdot \dot{\vec{q}}(t)$  singles out only the action of the force parallel to the trajectory, when evaluating the work. The perpendicular component of the force changes the direction of motion, but it does not perform work.

b) A force that is always acting perpendicular to the velocity, i.e., perpendicular to the path of the particle, does not perform any work,

$$W = \int \vec{F}(t) \cdot \dot{\vec{q}}(t) dt = \int 0 dt = 0$$

For some forces the work depends only on the initial and on the final point of the trajectory. They are called conservative forces because they can be used to define a total energy which is a constant of motion, i.e., it is conserved during the motion.

**Definition 3.5: Conservative Force**

A force  $\vec{F}$  is called conservative if it can be written as minus the gradient of a potential,  $\Phi(\vec{q})$

$$\vec{F}(\vec{q}) = -\nabla\Phi(\vec{q}) = - \begin{pmatrix} \partial_{q_1}\Phi \\ \vdots \\ \partial_{q_D}\Phi \end{pmatrix}$$

*Remark 3.3.* Conservative forces only depend on position,  $\vec{F} = \vec{F}(\vec{q})$ . They neither explicitly depend on time nor on the velocity  $\dot{\vec{q}}$ .

**Theorem 3.3: Work for conservative forces**

For conservative forces,  $\vec{F} = -\nabla\Phi(\vec{q})$ , the work for a path  $\vec{q}(t)$  from  $\vec{q}_0$  to  $\vec{q}_1$  amounts to the difference of the potential evaluated at the initial and at the final point of the path

$$W = \int_{\vec{q}(t)} \vec{F} \cdot d\vec{q} = \Phi(\vec{q}_0) - \Phi(\vec{q}_1)$$

*Proof.*

$$\begin{aligned} W &= \int_{t_0}^{t_1} \vec{F} \cdot \dot{\vec{q}} dt = - \int_{t_0}^{t_1} \nabla\Phi \cdot \dot{\vec{q}} dt = - \int_{t_0}^{t_1} \sum_i \frac{\partial\Phi}{\partial q_i} \frac{\partial q_i}{\partial t} dt \\ &= - \int_{t_0}^{t_1} \frac{d\Phi}{dt} dt = -(\Phi(\vec{q}(t_1)) - \Phi(\vec{q}(t_0))) = \Phi(\vec{q}_0) - \Phi(\vec{q}_1) \end{aligned}$$

□

*Remark 3.4.* The work performed along a closed path vanishes for conservative forces. After all, in that case  $\vec{q}_1 = \vec{q}_0$  such that  $W = \Phi(\vec{q}_0) - \Phi(\vec{q}_1) = 0$ .

**Theorem 3.4: Conservation of the total energy**

The *total energy*  $E = T + \Phi$  of a particle is conserved if it moves in a conservative force field  $\vec{F} = -\nabla\Phi$ .

*Proof.*

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{d\Phi}{dt} = m \dot{\vec{q}} \cdot \ddot{\vec{q}} + \nabla\Phi \cdot \dot{\vec{q}} = \dot{\vec{q}} \cdot \underbrace{(m\ddot{\vec{q}} - \vec{F})}_{= \vec{0}} = \vec{0}$$

In the third equality we used that the force is conservative, and in the final step, we used Newton's second law which states that  $m\ddot{\vec{q}} = \vec{F}$ .

□



## 3.5 Falling men and cat — a worked example

add worked example

## 3.6 Problems

## Rehearsing Concepts

**Problem 3.1. Derivatives of Elementary Functions**

Determine the derivatives of the following functions.

- |             |              |               |
|-------------|--------------|---------------|
| a) $\sin x$ | e) $\sinh x$ | i) $\ln x$    |
| b) $\cos x$ | f) $\cosh x$ | $\log_{10} x$ |
| c) $\tan x$ | g) $\tanh x$ | $n^x$         |
| d) $x^n$    | h) $e^x$     | $x^x$         |

**Problem 3.2. Integrals of Elementary Functions**

Evaluate the following integrals.

- |                              |                                   |  |
|------------------------------|-----------------------------------|--|
| a) $\int_{-1}^1 dx (a+x)^2$  | c) $\int_0^\infty dx e^{-x/L}$    | f) $\int_0^\infty dx x e^{-x^2/(2Dt)}$                 |
| b) $\int_{-5}^5 dq (a+bq^3)$ | d) $\int_{-L}^L dy e^{-y/\xi}$    | g) $\int_{-\sqrt{Dt}}^{\sqrt{Dt}} dl l e^{-l^2/(2Dt)}$ |
| $\int_0^B dk \tanh^2(kx)$    | e) $\int_0^L dz \frac{z}{a+bz^2}$ | $\int_{-\sqrt{Dt}}^{\sqrt{Dt}} dz x e^{-zx^2}$         |

Except for the integration variable all quantities are considered to be constant.

Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

**Problem 3.3. Collisions on a billiard table.**

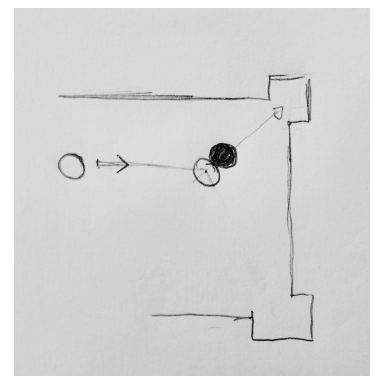
The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity  $\vec{v}$ ), and hit the black ball such that it ends up in pocket to the top right.

What is tricky about the sketched track?

What might be a better alternative?

**Problem 3.4. Pulling a Duck.**

A child is pulling a toy duck with a force of  $F = 5$  N. The duck has a mass of  $m = 100$  g and the chord has an angle  $\theta = \pi/5$  with the horizontal.<sup>3</sup>





<sup>3</sup> For this angle one has  $\tan \theta \approx 3/4$ .

- Describe the motion of the duck when there is no friction.  
In the beginning the duck is at rest.
- What changes when there is friction with a friction coefficient of  $\gamma = 0.2$ , i.e. a horizontal friction force of magnitude  $-\gamma mg$  acting on the duck.
- Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

### Practicing Concepts

#### Problem 3.5. Derivatives of Common Composite Expressions

Evaluate the following derivatives.

- $\frac{d}{dx}(a+x)^b$
- $\frac{\partial}{\partial x}(x+by)^2$
- $\frac{d}{dx}(x+y(x))^2$
- $\frac{d}{dt} \sin \theta(t)$
- $\frac{d}{dt}(\sin \theta(t) \cos \theta(t))$
- $\frac{d}{dt} \sin(2\theta(t))$
- $\frac{d}{dz} \sqrt{a+bz^2}$
- $\frac{\partial}{\partial x_3} \left[ \sum_{j=1}^6 x_j^2 \right]^{-1/2}$
- $\frac{\partial}{\partial y_1} \ln(\vec{x} \cdot \vec{y})$

In these expressions  $a$  and  $b$  are real constants, and  $\vec{x}$  and  $\vec{y}$  are 6-dimensional vectors.

#### Problem 3.6. Solving Integrals by Partial Integration

Evaluate the following integrals by partial integration

$$\int dx f(x) g'(x) = f(x) g(x) - \int dx f'(x) g(x)$$

- $\int_a^b dx x e^{kx}$
- $\int_a^b dx x^2 e^{kx}$
- $\int_a^b dx x^n e^{kx},$   
 $n \in \mathbb{N}$

The integral c) can only be given as a sum over  $j = 0, \dots, n$ .

#### Problem 3.7. Substitution with Trigonometric and Hyperbolic Functions

Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{q(x_1)}^{q(x_2)} dq f(q) = \int_{x_1}^{x_2} dx q'(x) f(q(x))$$

with a function  $q(x)$  that is bijective on the integration interval  $[x_1, x_2]$ .

- $\int_a^b dx \frac{1}{\sqrt{1-x^2}}$  by substituting  $x = \sin \theta$

- b)  $\int_a^b dx \frac{1}{\sqrt{1+x^2}}$  by substituting  $x = \sinh z$
- c)  $\int_a^b dx \frac{1}{1+x^2}$  by substituting  $x = \tan \theta$
- d)  $\int_a^b dx \frac{1}{1-x^2}$  by substituting  $x = \tanh z$

**Problem 3.8. Crossing a River.**

A ferry is towed at the bank of a river of width  $B = 100$  m that is flowing at a velocity  $v_F = 4$  m/s to the right. At time  $t = 0$  s it departs and is heading with a constant velocity  $v_B = 10$  km/h to the opposite bank.

- a) When will it arrive at the other bank when it always heads straight to the other side? (In other words, at any time its velocity is perpendicular to the river bank.)

How far will it drift downstream on its journey?

- b) In which direction (i.e. angle of velocity relative to the downstream velocity of the river) must the ferryman head to reach exactly at the opposite side of the river?

Determine first the general solution. What happens when you try to evaluate it for the given velocities?

**Problem 3.9. Retroreflector paths on bike wheels.**

The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retroreflectors fixed in the sparks enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio  $\rho$  of the reflectors distance  $d$  to the wheel axis and the wheel radius  $r$ . A small stone in the profile traces a cycloid ( $\rho = 1$ ). Animations of the trajectories can be found at

<https://en.wikipedia.org/wiki/Trochoid> and <http://katgym.by.lo-net2.de/c.wolfseher/web/zykloiden/zykloiden.html>.

A trochoid is most easily described in two steps: Let  $\vec{M}(\theta)$  be the position of the center of the disk, and  $\vec{D}(\theta)$  the vector from the center to the position  $\vec{q}(\theta)$  that we follow (i.e. the position of the retroreflector) such that  $\vec{q}(\theta) = \vec{M}(\theta) + \vec{D}(\theta)$ .

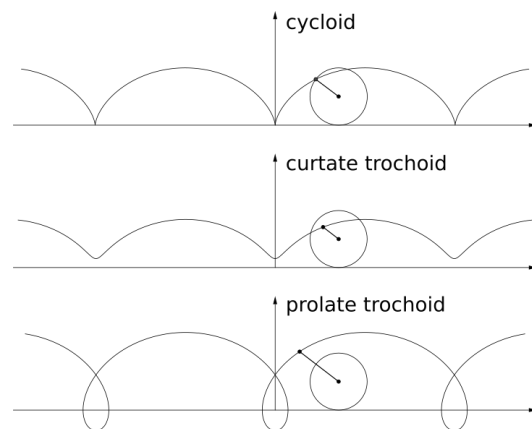


Figure 3.4: [Wikimedia CC BY 4.0 (modified)]

- a) The point of contact of the wheel with the street at the initial time  $t_0$  is the origin of the coordinate system. Moreover, we single out one spark and denote the change of its angle with respect to its initial position as  $\theta$ . Note that negative angles  $\theta$  describe forward motion of the wheel!

Sketch the setup and show that

$$\vec{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix}, \quad \vec{D}(\theta) = \begin{pmatrix} -d \sin(\varphi + \theta) \\ d \cos(\varphi + \theta) \end{pmatrix}.$$

check signs of components of  $\vec{D}$

What is the meaning of  $\varphi$  in this equation?

- b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time,  $L = \int_{t_0}^t dt |\dot{\vec{q}}(\theta(t))|$ . Show that therefore

$$L = r \int_0^\theta d\theta \sqrt{1 + \rho^2 + 2\rho \cos(\varphi + \theta)}$$

- c) Consider now the case of a cycloid and use  $\cos(2x) = \cos^2 x - \sin^2 x$  to show that the expression for  $L$  can then be written as

$$L = 2r \int_0^\theta d\theta \left| \cos \frac{\varphi + \theta}{2} \right|$$

How long is one period of the track traced out by a stone picked up by the wheel profile?

*Proofs*

*Transfer and Bonus Problems, Riddles*

**Problem 3.10. Running Mothers.**

In the Clara Zetkin Park one regularly encounters blessings<sup>4</sup> of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their young in pouches.

- a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.  
The carriages suffer from friction. Let the friction coefficient be  $\gamma = 0.3$ .  
When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.
- b) How does the running speed matter in this discussion?

<sup>4</sup>Look up "terms of vengery" if you ever run out of collective nouns.

- c) How does the mass of the babies/young make a difference?

**Problem 3.11. Hypotrochoids, roulettes, and the spirograph.**

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a **spirograph**, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

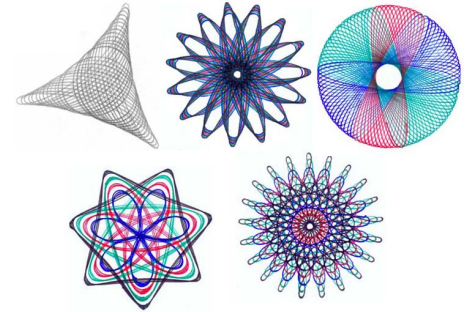


Figure 3.5: [Wikimedia Public domain]

- a) Consider the track of a pole attached to a disk with  $n$  cogs that rolls inside a circular curve with  $m > n$  cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)
- b) Adapt the description for the curves developed in Problem 3.9 such that you can describe hypotrochoids.
- c) Test your result by writing a Python program that plots the cruves for given  $m$  and  $n$ .



# 4

## *Equations of Motion*

In Chapter 3 we learned how to set up a physical model based on finding the forces acting on a body, and thus determining the acceleration of its motion. For a particle of mass  $m$  and position  $\vec{q}$  resulting equation relates its acceleration  $\ddot{\vec{q}}$  to the force, that in itself may depend on  $\vec{q}$ ,  $\dot{\vec{q}}$ , and  $t$ . The resulting equation

$$m \ddot{\vec{q}} = \vec{F}(\vec{q}, \dot{\vec{q}}, t) \quad (4.0.1)$$

is referred to as the *equation of motion* (EOM) of the particle. From the mathematical point of view it is an *ordinary differential equation* (ODE). The present chapter will

- i. introduce ODEs,
- ii. discuss strategies to characterize sets of solutions and find solutions for specific initial conditions, and
- iii. discuss the solutions of examples that are of particular relevance in physics.

In the end we will be able to discuss the motion of planets around the sun and moons around their planets — as well as the fate of a that lost his electromagnetic interaction with Earth.

### *4.1 Motivation and Outline: What are EOMs and ODEs?*

The *order* of a differential equation denotes the highest order derivative that appears in the equation. It is called an *ordinary differential equation*, when all derivatives are taken with respect to the same variable. From this perspective Equation (4.0.1) is a ordinary differential equation (ODE) of second order. It is called *autonomous* when its right-hand side does not explicitly depend on time.

For  $N$  particles with masses  $m_i$ ,  $i = 1 \cdots N$  that are moving in  $D$  dimensions we will have a system of  $N$  differential equations for the  $D$  dimensional vectors  $\vec{q}_i = (q_{i,\alpha}, \alpha = 1 \cdots D)$

$$\ddot{q}_{i,\alpha} = \frac{1}{m_i} F_{i,\alpha}(\{\vec{q}_i, \dot{\vec{q}}_i\}_{i=1 \cdots N}, t), \quad i = 1 \cdots N, \quad \alpha = 1 \cdots D$$

By introducing the variables  $\vec{v}_i = \dot{\vec{q}}_i$  the EOMs can always be written as a set of  $2N$  first order ODE

$$\begin{aligned} \dot{q}_{i,\alpha} &= v_{i,\alpha} \\ \dot{v}_{i,\alpha} &= \frac{1}{m_i} F_{i,\alpha}(\{\vec{q}_i, \vec{v}_i\}_{i=1 \cdots N}, t) \end{aligned}$$

For an autonomous system this can be written in a more compact form by introducing the  $2DN$  dimensional phase-space coordinate  $\vec{\Gamma}$  and the flow  $\vec{\mathcal{V}}$  as follows

$$\begin{aligned} \vec{\Gamma} &= (q_{1,1} \cdots q_{1,D}, q_{2,1} \cdots q_{N,D}, \dot{q}_{1,1} \cdots \dot{q}_{1,D}, \dot{q}_{2,1} \cdots \dot{q}_{N,D}) \\ \vec{\mathcal{V}} &= \left( v_{1,1} \cdots v_{1,D}, v_{2,1} \cdots v_{N,D}, \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}, \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N} \right) \\ \dot{\vec{\Gamma}} &= \vec{\mathcal{V}}(\vec{\Gamma}) \quad \text{for autonomous systems.} \end{aligned}$$

Moreover, a non-autonomous system can always be expressed as an autonomous, first order ODE where  $\vec{\gamma}$  and  $\vec{\mathcal{V}}$  denote points in a  $2DN + 1$  dimensional phase space, however,

$$\begin{aligned} \vec{\Gamma} &= (q_{1,1} \cdots q_{1,D}, q_{2,1} \cdots q_{N,D}, \dot{q}_{1,1} \cdots \dot{q}_{1,D}, \dot{q}_{2,1} \cdots \dot{q}_{N,D}, t) \\ \vec{\mathcal{V}} &= \left( v_{1,1} \cdots v_{1,D}, v_{2,1} \cdots v_{N,D}, \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}, \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}, 1 \right) \\ \dot{\vec{\Gamma}} &= \vec{\mathcal{V}}(\vec{\Gamma}) \quad \text{for non-autonomous systems.} \end{aligned}$$

In phase space,  $\Gamma$  denotes a point that characterizes the state of our system, and  $\vec{\mathcal{V}}(\Gamma)$  provides the *unique* direction and velocity of the temporal change of this state. In a crude approximation, that is accurate however for sufficiently small  $\Delta t$ , we have

$$\vec{\Gamma}(t + \Delta t) \simeq \vec{\Gamma}(t) + \Delta t \vec{\mathcal{V}}(\vec{\Gamma}(t))$$

This admits a graphical representation of the solutions of the ODE that we will explore in the next section when we discuss the solutions of some ODEs that commonly arise in mechanical problems.



## 4.2 Free flight

We first discuss the motion of a single particle that is moving in a gravitational field giving rise to the constant gravitational acceleration  $\vec{g}$ . Hence, the particle position  $\vec{q}(t)$  obeys the EOM

$$\ddot{\vec{q}} = \vec{g} \quad (4.2.1)$$

The right hand side of this equation does not depend on  $\vec{q}$ . This has two remarkable consequences that we will exploit whenever possible.

### *Decoupling of the motion of different DOF*

Each component  $q_\alpha$  of  $\vec{q}$  can be solved independently of the other DOF

$$\dot{q}_\alpha = g_\alpha$$

Rather than dealing with a vector-valued ODE, one can therefore solve  $D$  scalar ODEs which turns out to be a much simpler task. Indeed, we will see in our further discussion that the solution of vector-valued ODEs will often proceed via a coordinate transformation that decouples the different DOF.

### *The ODE can be integrated*

The ODE, Equation (4.2.1), can be solved by integration

#### Algorithm 4.1: Integrating ODEs

When the right-hand side of a differential equation  $\ddot{\vec{f}}(t) = \vec{v}(t)$  does not depend on  $\vec{q}$  then it can be solved by integration. For the initial condition  $\vec{f}(t_0) = \vec{f}_0$  one thus obtains

$$\vec{f}(t) = \vec{f}_0 + \int_{t_0}^t dt' \dot{\vec{f}}(t') = \vec{f}_0 + \int_{t_0}^t dt' \vec{v}(t')$$

Moreover, when the ODE is autonomous we obtain

$$\vec{f}(t) = \vec{f}_0 + \vec{v}(t - t_0)$$

For the free flight we thus obtain for an initial conditions  $\vec{q}(t_0) =$

$\vec{q}_0$  and  $\dot{\vec{q}}(t_0) = \vec{v}_0$ , that

$$\dot{\vec{q}}(t) = \vec{v}_0 + \int_{t_0}^t dt' \vec{g} = \vec{v}_0 + \vec{g} (t - t_0)$$

and

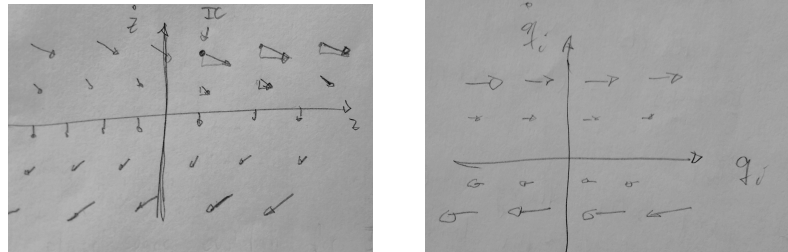
$$\begin{aligned} \vec{q}(t) &= \vec{q}_0 + \int_{t_0}^t dt' \dot{\vec{q}}(t') = \vec{q}_0 + \int_{t_0}^t dt' (\vec{v}_0 + \vec{g} (t' - t_0)) \\ &= \vec{q}_0 + \vec{v}_0 \int_{t_0}^t dt' + \vec{g} \int_{t_0}^t dt' (t' - t_0) \\ &= \vec{q}_0 + \vec{v}_0 (t - t_0) + \vec{g} \int_0^{t-t_0} dt'' (t - t_0) \\ &= \vec{q}_0 + \vec{v}_0 (t - t_0) + \frac{1}{2} \vec{g} (t - t_0)^2 \end{aligned}$$

When we denote the direction antiparallel to  $\vec{g}$  as  $z = q_1$ , then

$$\begin{aligned} z(t) &= q_1(t) = z(t_0) + v_z(t_0) (t - t_0) - \frac{g}{2} (t - t_0)^2 \\ q_i(t) &= q_i(t_0) + v_i(t_0) (t - t_0), \quad \text{for } i > 1 \end{aligned}$$

In phase space these equations are represented by the flow Non-

Figure 4.1: Phase-space flow for motion with constant acceleration (left) and for free flight (right).



dimensionalization leads in this case to another interesting finding.

Let us measure positions  $q_i(t)$  in multiples of  $q_i(t_0)$ , velocities in multiples of  $v_i(t_0)$ , and introduce the dimensionless time  $\tau = v_i(t_0)(t - t_0)/q_i(t_0)$ . Then the solution of the EOM take the form

$$\begin{aligned} \hat{z}(\tau) &= \frac{q_1(t)}{q_1(t_0)} = 1 + \tau - I \tau^2, \quad \text{with } I = \frac{g q_1(t_0)}{2 v_i^2(t_0)} \\ \hat{q}_i(t) &= \frac{q_i(t)}{q_i(t_0)} = 1 + \tau, \quad \text{for } i > 1 \end{aligned}$$

The second relation implies that in phase space  $\hat{q}_i$  is a straight horizontal line through  $\hat{v}_i = 1$ , and from the first equation we find  $\tau = (1 - v)/2I$  such that all trajectories in phase space are

parabola of the form

$$\hat{v}_z^2 = 1 - 4I(\hat{z} - 1) \quad (4.2.2)$$

#### 4.3 Free flight with Newtonian friction

add worked axample

#### 4.4 Free flight with turbulent friction

add worked axample

#### 4.5 Particle suspended from a spring

add worked axample

#### 4.6 Kepler's laws for planetary motion — a worked example

add worked axample

#### 4.7 Falling through Earth — a worked example

add worked axample

#### 4.8 Problems

##### Rehearsing Concepts

##### Problem 4.1. Phase-space portraits for a scattering problem.

Sketch the potential  $\Phi(x) = 1 - 1/\cosh x$  for  $x \in \mathbb{R}$ . Add to the sketch a the phase portait of the motion in this potential, i.e. the solutions of  $\ddot{x} = -\partial_x \Phi(x)$ , in the phase space  $(x, \dot{x})$ .

##### Problem 4.2. Phase-space portraits for the Kepler and the DLVO problem.

The figures below show the effective potentials for the distance between two planets in the Kepler problem, and for the DLVO potential for the interaction of charged colloids.<sup>1</sup> Sketch the solutions for classical trajectories in these potentials in the phase space  $(R, \dot{R})$ .

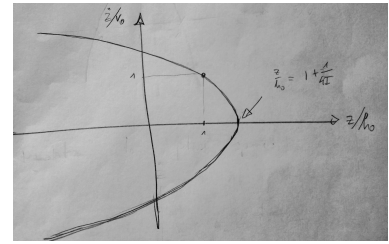
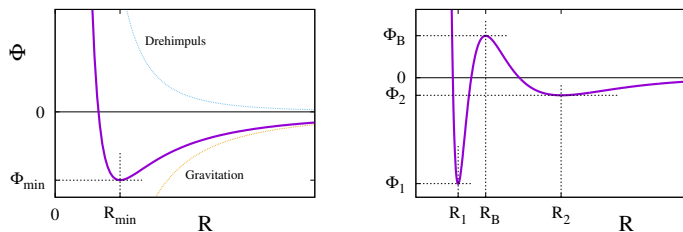


Figure 4.2: Sketch of the universal form of the free-flight trajectories in phase space, Equation (4.2.2).

<sup>1</sup> The DLVO theory predicts that there are two distinct average bond length for aggregates of two colloids. There is a strong bond of strength  $\Phi_1$  where the colloids have a small bond length  $R_1$ , and a weak bond of strength  $\Phi_2$  at a larger distance  $R_2$ . Between these two states there is an energy barrier of height  $\Phi_B$ .



**Problem 4.3. Gradients and equipotential lines**

Determine the derivatives of the following functions.

- a) Equipotential lines in the  $(x, y)$ -plane are lines  $y(x)$  or  $y(x)$  where a functions  $f(x, y)$  takes a constant value. Sketch the equipotential lines of the functions

$$f_1(x, y) = (x^2 + y^2)^{-1} \quad \text{and} \quad f_2(x, y) = -x^2 y^2$$

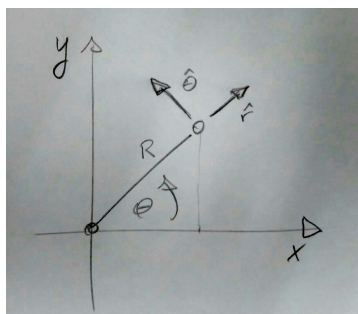
- b) Determine the gradients  $\nabla f_1(x, y)$  and  $\nabla f_2(x, y)$ .  
Hint: The gradient  $\nabla f_i(x, y)$  with  $i \in \{1, 2\}$  is a vector  $(\partial_x f_i(x, y), \partial_y f_i(x, y))$  that contains the two partial derivatives of the (scalar) function  $f_i(x, y)$ .
- c) Indicate the direction and magnitude of the gradient by appropriate arrows in the sketch showing the equipotential lines. In which direction is the gradient pointing?

*Practicing Concepts*

**Problem 4.4. Motion on a circular track.**

The position of a particle in the plane can be specified by Cartesian coordinates  $(x, y)$  or by polar coordinates with basis vectors  $\hat{r}(\theta)$  and  $\hat{\theta}(\theta)$ , that have the following representation in Cartesian coordinates (cf. the sketch to the left)

$$\hat{r} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \text{and} \quad \hat{\theta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$



We will now explore the trajectory  $\vec{q}(t)$  of a particle with mass  $m$  that moves on a track with a fixed radius  $R$ .

- a) Verify that  $\hat{\theta} = \partial_\theta \hat{r}$  and  $\partial_\theta^2 \hat{r} = -\hat{r}$ .  
Please provide a geometric interpretation of this result!
- b) The position of the particle can be specified as  $\vec{q}(t) = R \hat{r}(\theta(t))$ . Determine  $\dot{\vec{q}}$  and  $\ddot{\vec{q}}$  based on this equations. Verify your result by performing the same calculation in Cartesian coordinates.

- c) Which force is required to keep the particle on the circular track? What does this imply for curves in **bike races**, **bobsled races** and in **skate parks**?
- d) Consider the motion at a constant angular velocity,  $\theta(t) = \omega t$ , and show that the acceleration in this setting takes the form  $\ddot{\vec{q}} = -R\omega^2 \hat{r}(\omega t)$ . Verify that this amounts to a force that is perpendicular to the velocity. What does this imply for the absolute value of the velocity?  
Hint: Discuss the time derivative of  $\vec{v}^2$ .

**Problem 4.5. Free fall with viscous friction.**

The falling of a ball in a viscous medium can be described by the equations of motion

$$\ddot{h}(t) = -g - \gamma \dot{h}(t).$$

Here  $h(t)$  is the vertical position of the ball,  $g$  is the acceleration due to gravity, and the coefficient  $\gamma \simeq \frac{2}{3}\eta/\rho_{\text{ball}}R^2$  describes the viscous drag. Here  $R$  is the radius of the sphere,  $\rho_{\text{ball}}$  is the mass density of its material, and  $\eta$  is the viscosity of the surrounding fluid. For air and water it takes values of about  $\eta_{\text{air}} \simeq 2 \times 10^{-5} \text{ kg/m s}$ , and  $\eta_{\text{water}} \simeq 1 \times 10^{-3} \text{ kg/m s}$ , respectively.

- a) Argue that  $w(\tau) = \gamma \dot{h}(t)/g$  with  $\tau = \gamma t$  obeys the equation

$$\frac{dw(\tau)}{d\tau} = -1 - w(\tau).$$

How do you recover the the dependence of the motion on the parameters  $g$  and  $\gamma$ ?

- b) Determine the solution of the equation for the initial condition  $w(\tau_0) = w_0$ . What happens for  $w_0 = -1$ ?
- c) Determine  $h(t)$  for a ball that is released at a height  $H$  with zero velocity, and with an upward velocity of  $v_0$ .
- d) Sketch the solution  $h(t)$ . How does the solution look like for small and for large  $t$ ?  
In particular: Determine the Taylor expansion for the trajectory to third order in  $t$ . How does it differ from a free fall with  $\gamma = 0$ ?
- e) Estimate the time scale where the viscosity does not yet lead to noticeable differences from a description with viscosity of a

bullet with a diameter of 1 cm, when it drops down from the balcony and when it is vertically shot into the air with initial velocity 100 m/s. How far did it travel in that time?

- f) How do the conclusions change for a harpoon shot under water? (Assume for simplicity that it is sufficient to treat it like a ball with radius corresponding to the diameter of the arrow.)

**Problem 4.6. Free fall with turbulent friction.**

For large velocities the motion of the fluid around the ball goes turbulent, and the friction crosses over to a drag force with modulus

$$F_D = \frac{\rho_{\text{fluid}} \pi R^2 C_D}{2} v^2 = m \kappa v^2$$

where  $m$  is the mass of the ball and  $C_c$  is a drag coefficient that typically takes values between 0.5 and 1. A very beautiful description of the physics of this equations has been provided in an instruction video by the NASA (click [here](#) to check it out).

- a) Argue that  $u(\tau) = \sqrt{\kappa/mg} \dot{h}(t)$  with  $\tau = \sqrt{\kappa/mg} t$  obeys the equation

$$\frac{du(\tau)}{d\tau} = -1 - u^2(\tau) \text{sign}(u(\tau)).$$

How do you recover the dependence of the motion on the parameters  $g$  and  $\kappa$ ?

- b) Determine the solution of the equation for the initial condition  $u(\tau_0) = u_0$ . What happens for  $u_0 = -1$ ?<sup>2</sup>
- c) Determine  $h(t)$  for a ball that is released at a height  $H$  with zero velocity, and with an upward velocity of  $v_0$ .
- d) Do you trust the solution for zero velocity? If not: provide an estimate for the range of parameters and initial conditions where the solution applies.

<sup>2</sup> **Hint:** The sign-function is best handled as follows. Start by dealing with initial conditions where  $u_0 \leq 0$ , and show that the velocity will then always remain negative. In this case  $\text{sign}(u(\tau)) = -1$  for all times. Subsequently, explore initial conditions with  $u_0 > 0$ . For these initial conditions you look for solutions where  $u(\tau) > 0$ . At some time  $\tau_*$  and position  $z_*$  the velocity becomes negative, however. The trajectory will then continue according to a trajectory started at time  $\tau_*$  at position  $z_*$  with velocity zero. This evolution we know, however.

**Problem 4.7. Determining the volume, the mass, and the center of mass**

Determine the mass  $M$ , the area or volume  $V$ , and the the center of mass  $\vec{Q}$  of bodies with the following mass density and shape.

- a) A triangle in two dimensions with constant mass density  $\rho = 1 \text{ kg/m}^2$  and side length 6 cm, 8 cm, and 10 cm.

Hint: Determine first the angles at the corners of the triangle. Decide then about a convenient choice of the coordinate system (position of the origin and direction of the coordinate axes).

- b) A circle in two dimensions with center at position  $(a, b)$ , radius  $R = 5 \text{ cm}$ , and constant mass density  $\rho = 1 \text{ kg/m}^2$ .  
Hint: How do  $M$ ,  $V$  and  $\vec{Q}$  depend on the choice of the origin of the coordinate system?
- c) A rectangle in two dimensions, parameterized by coordinates  $0 \leq x \leq W$  and  $0 \leq y \leq B$ , and a mass density  $\rho(x, y) = \alpha x$ .  
What is the dimension of  $\alpha$  in this case?
- d) A three-dimensional wedge with constant mass density  $\rho = 1 \text{ kg/m}^3$  that is parameterized by  $0 \leq x \leq W$ ,  $0 \leq y \leq B$ , and  $0 \leq z \leq H - Hx/W$ .  
Discuss the relation to the result of part b).
- e) A cube with edge length  $L$ . When its axes are aligned parallel to the axes  $\hat{x}, \hat{y}, \hat{z}$ , its density takes the form  $\rho(x, y, z) = \beta z$ .  
What is the dimension of  $\beta$  in this case?

### Proofs

#### Transfer and Bonus Problems, Riddles

#### Problem 4.8. Maximum distance of flight.

There is a well-known rule that one should throw a ball at an angle of roughly  $\theta = \pi/4$  to achieve a maximum width.

- a) Solve the equation of motion of the ball thrown in  $x$  direction with another velocity component in vertical  $z$  direction. Do not consider friction in this discussion, and verify that the ball will then proceed on a parabolic trajectory in the  $(x, z)$  plane.
- b) Well-trained shot put pushers push the put with an initial angle substantially smaller than  $\pi/4$ , i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height  $H$  of the release point of the trajectory over the ground is noticeable as compared to the length  $L$  between the release point and touchdown, i.e. when  $H/L$  is not small.

Challenge. What is the optimum choice of  $\theta$  for the shot put?

- c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?



## *Take Home Message*

### *Hints for Exam Preparation*

The aim of the present course has been to give a first glimpse into scientific modeling. It focussed on mechanics problems. Firstly, they are easily visualized. Secondly, they readily provide interesting mathematical challenges when one strives for a comprehensive description. Thus, they provide a unique set of problems to get acquainted with the use of mathematics as a language to address scientific problems. The involved mathematical concepts will further be underpinned in forthcoming mathematics classes. Further physical problems will be addressed in experimental and theoretical physics lectures.

What are the next steps to be taken? To begin with you should re-read the script and revisit the exercise sheets in order to prepare for the exam. Take a particular look at exercises that were challenging at the first encounter. In doing so you should focus on understanding the rules of the game, and hands-on application of the mathematical formalism, rather than understanding the concepts in full depth. The concepts will be dealt with again in mathematics classes, that will *not* address, however, practicalities about efficient use in concrete calculations.

**Best wishes, success and fun for your further studies!**



## *Todo list*

---

The Todo List is empty!  
Why?