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Sommersemester 2019
Fr, 5. 7. 2019

Pattern Formation and Nonlinear Dynamics

14. Water Waves

1. Estimates.

The dispersion relation of water entails some remarkably accurate estimates of distances, speed, and wave patterns.

(a) In the middle of the ocean a storm generates waves. After a while they arrive at the next coast with a period of 15 sec, and 24 hours later this period has dropped to 12 sec. Estimate the distance between the center of the storm and coast.

(b) Yachtsmen estimate the maximum speed $v_{\max}$ of a vessel by the rule

$$v_{\max} = \frac{4}{3}\sqrt{L}\,,$$

where L is the length of the boat given in feet and $v_{\max}$ is expressed in knots. Provide a physical derivation of this rule.

(c) A stream in a river can be identified by standing capillary waves upstream, and standing gravitational waves downstream. What is the relation between their wave length and the streaming velocity of the river?

3. Ship Wakes.

Every swimmer eject waves in his/her/its wake. They are bounded to the left and right by feather waves, and terminate on a well-defined angle θ. The picture to the left shows the emerging wave patter for a duck. In this exercise we explore the differences and similarities of the waves to the Mach cone ejected in supersonic flight.

Daderot [Public domain]

(a) In the atmosphere sound travels at a fixed velocity c. When an object is flying with a velocity $v > c$ the sound is therefore confined in a cone. Verify the the opening angle φ of this cone obeys the relation

$$\sin\varphi = \frac{c}{v}$$

such that the speed of flight can be inferred from the opening angle, $v = c/\sin\varphi$.

(b) Let us now consider a ship (or duck) that is moving with speed U. It emits surface water waves with wave length up to its water length. Again the respective wave packages are confined in a conical region. However, now the propagation speed c depends on λ such that

$$\varphi = \sup_\lambda \arcsin \frac{c(\lambda)}{v}$$

How does λ depend on U?

3. Latte Macchiato.

A Latte Macchiato consists of hot coffee that is carefully (through a foam) positioned on top of a layer of much colder milk. The surface tension is negligible, but there is some difference in density. After taking a sip and carefully setting back the glass, this can be be noticed by beautiful gravity waves of the internal interface. The period allows us to estimate this density difference. For simplicity we assume that the glass has a quadratic cross section, and that the flow is two dimensional, deriving from a potential $\phi(x, y, t)$.

(a) In the lecture we derived surface waves by only considering the flow below the surface. This must be generalized now to account also for the flow of the fluid above the interface. Let the densities in the upper and lower layer be ρ_u and ρ_d, respectively, and show

$$\rho_u \, \left(\partial_t \phi + \frac{\vec{u}^2}{2} - g\,\eta \right) + P = G_u(t)$$
$$\rho_d \, \left(\partial_t \phi + \frac{\vec{u}^2}{2} + g\,\eta \right) + P = G_d(t)$$
$$\partial \eta + \partial_y \phi + \partial \phi \, \partial \eta = 0$$

(b) Eliminate P by the choice of $G_i(t)$, linearize the equations, and derive the dispersion relation

$$c^2 = \frac{g}{k} \, \frac{\rho_d - \rho_u}{\rho_d + \rho_u}$$

(c) What does the result tell about the density difference of coffee and milk?

4. Rayleigh-Taylor instability.

For the Latte Macchiato the density of the upper fluid was larger than that of the lower one, $\rho_d > \rho_u$. The Rayleigh-Taylor problem deals with the reverse situation where $\rho_u > \rho_d$. Otherwise, the setting is the same.

(a) Show that the system is always linearly unstable when there is no surface tension.

(b) Show that surface tension can stabilize the system if the lateral dimension L is small. Find the dimensionless control parameter for this problem, and determine the critical value where the system become linearly unstable. What does linear instability actually mean for a system with a second derivative in time? What type of instability do we encounter?

** **Bonus Problems.**

(a) Make a photo of a shallow layer of water that is running down a street, e.g. immediately after it was raining. Measure the wave length of the waves on its surface, and compare your measurement to the prediction of the linear-stability analysis.

(b) You will see ripples forming on you tea when you gently blow over a cup with hot soup. Assume that the air is a fluid which is running with constant velocity U over the fluid, and perform the linear-stability analysis of the plane surface.