Universität Leipzig
Fakultät für Physik und Geowissenschaften
Jürgen Vollmer

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Pattern Formation and Nonlinear Dynamics

13. Hydrodynamic Instabilities

1. Competition of Stripes.

We consider an isotropic 2D system whose stationary stripe patterns are described by an amplitude equation with cubic nonlinear terms of the form $|A|A$. In the simplest setting a patterns that involves a superposition of two plane waves with relative orientation θ is then described by

$$\tau_0 \partial_t A_1 = \epsilon\, A_1 - g_0 \left(|A_1|^2 + G(\theta)|A_2|^2 \right) A_1$$
$$\tau_0 \partial_t A_2 = \epsilon\, A_2 - g_0 \left(|A_2|^2 + G(\theta)|A_1|^2 \right) A_2$$

The equations are related by permutation of the indices.

(a) The stripe-coupling coefficient $G(\theta)$ has the symmetry $G(\theta) = G(\pi - \theta)$. Do you see why this is necessary?

(b) Determine the amplitude A_S of a stationary stripe pattern where $A_S = |A_1|$ and $|A_2| = 0$.
Discuss the linear stability of this state towards perturbations where a second stripe appears at an angle θ. How does the result depend on $G(\theta)$?

(c) Determine the amplitude A_L of a stationary lattice pattern where $A_L = |A_1| = |A_2|$.
Discuss the linear stability of this state towards perturbations where one of the stripe decays. How does the result depend on $G(\theta)$?

(d) For the 2D Swift-Hohenberg equation the stripe-coupling coefficient takes the constant values $G(\theta) = 2$. What does this imply for the observed patterns. Assume that your system admits a small perturbation that will lead to an non-trivial θ-dependence of $G(\theta)$. How does this impact the observed patterns?

2. Stripe coupling in the Swift-Hohenberg model.

In this exercise we determine the stripe-coupling coefficient $G(\theta)$ of the isotropic Swift-Hohenberg model

$$\partial_t u(\vec{x},t) = r\, u(\vec{x},t) - \left(1+\nabla^2\right)^2 u(\vec{x},t) - u^2(\vec{x},t)$$
$$\text{with} \quad \vec{x} = (x_1, x_2) \quad \text{and} \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right).$$

(a) Consider an ansatz of the form

$$u_L(\vec{x},t) = a_L(\theta)\ [\cos(\vec{q}_1 \cdot \vec{x}) + \cos(\vec{q}_1 \cdot \vec{x})] \qquad \text{with} \quad |\vec{q}_1| = |\vec{q}_2| = 1 \quad \text{and} \quad \vec{q}_1 \cdot \vec{q}_2 = \cos\theta$$
$$u_S(\vec{x},t) = a_S \cos(\vec{q}_s \cdot \vec{x}) \qquad \text{with} \quad |\vec{q}_S| = 1$$

for a lattice and stripe state, respectively. For which amplitudes $a_L(\theta)$ and a_S are these stationary solutions of the Swift-Hohenberg equation?

(b) Determine the stripe-coupling coefficient $G(\theta)$ by compare $a_L(\theta)$ and a_S to A_L and A_S of exercise 1.

3. The Dripping Faucet.

Have you ever wondered why faucets are dripping? There is a thin jet of water coming out of the faucet, and 20 cm further down there are droplets! The reason can be found by the following linear stability analysis.

(a) Consider a cylindrical water jet of diameter R_0. What is its surface area and volume per unit length?

(b) Let us do a linear stability analysis. To this end we consider a jet with cross section $R_1 + A\mathrm{e}^{\mathrm{i}kx+\sigma t}$ that experiences an undulation of wave length $k \in \mathbb{R}$. The imaginary part of $\sigma \in \mathbb{C}$ will account for traveling waves. Determine the surface area and volume per unit length for this jet.

(c) We will require that the volume per unit length is constant, when averaged over one period of the perturbation. What does that imply for the relation between R_1 and A?

(d) Due to surface tension the free energy per unit length $\mathcal{F}$ of the jet is proportional to its surface area per unit length. What does this imply for the stability of the jet? How is the stability effected by the the wave number k?

(e) The free energy is a function of the amplitude and the wavelength, $\mathcal{F} = \mathcal{F}(k, A)$. Consider now a steepest descent dynamics,

$$\begin{pmatrix} \dot{k} \\ \dot{A} \end{pmatrix} = -\gamma\ \nabla\mathcal{F}(k, A)\,.$$

Write a program that follows a set of trajectories with initial conditions $(k, A) = (k_0, 0)$ in the (k, A) plane. Where do they go? Which physical constraints do you have to imply on A? What does this tell about droplet breakup?

4. Water Waves.

Let us consider a fluid with flow velocity $\vec{u} = \big(u(x,y,t), v(x,y,t)\big)$. Gravity is acting in y-direction. Under suitable conditions, that will be discussed in the lectures next week, the flow can be derived from a potential $\vec{u} = \nabla\phi(x,y,t)$. The incompressibility condition $\nabla \cdot \vec{u} = 0$ implies that ϕ obeys the Laplace equation $\nabla^2\phi = 0$.

For gravity acting in y direction the evolution of an interface of the fluid at height $y = \eta(x,t)$ is described by the equations

$$\begin{aligned} \partial_y \phi\big|_{y=0} - \partial_t \eta &= 0 \\ \partial_t \phi\big|_{y=0} + g\,\eta - \frac{T}{\rho}\partial_x^2 \eta &= 0 \end{aligned}$$

where g is the gravitational acceleration and T the surface tension force.

(a) Consider a sinusoidal traveling wave

$$\eta = A\cos(kx - \omega t)\phi = f(y)\ \sin(kx - \omega t)$$

and determine $f(y)$ by solving the Laplace equation. How does the solution look like for infinitely deep water? How for shallow water with a depth smaller than the wave length?

(b) Determine the dispersion relation $\omega = \omega(k)$ for the surface waves. The wave velocity $c = \omega/k$ has a minimum as function k. What does this imply of the *standing* waves generated by an obstacle in a flow?