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Pattern Formation and Nonlinear Dynamics

12. Linear Stability and Amplitude Equations

1. Kuramoto-Sivashinsky Model.

The Kuramoto-Sivashinsky model

$$\partial_t u + u\,\partial_x u = -r\,\partial_x^2 u - \partial_x^4 u \qquad \text{for a scalar real field } u(x,t)$$

is one of the simplest models with a nonlinear term $u\,\partial_x u$. It has widely been studied as a model featuring spatio-temporal chaos. Here we discuss its linear stability and finite-size effects.

(a) Show that $u = 0$ is a solution for all values of r. This will be our reference solution for the linear stability analysis.

(b) Adopt an ansatz of the form $u \propto \mathrm{e}^{\gamma\, t + \mathrm{i}kx}$, and show that these perturbation are stable for $r < 0$, while some wavelength are unstable for $r > 0$. Which type of instability does the model show?

(c) Show that the modulus of r can be suppressed in the equation by a suitable choice of dimensionless units. Determine the resulting equations of motion for positive and negative values of r.

(d) Consider now finite domains with (i) periodic boundary conditions with period L, and (ii) a finite domain of size L with absorbing boundary conditions $u = \partial_x u = 0$. Show that in this situation $p = \sqrt{r}\,L$ will take the role of an order parameter. How do the critical values $p_c^{(i)}$ and $p_c^{(ii)}$ where the system passes for stable to unstable depend on the boundary condition?

2. Multiple Scale Analysis of a Generalized Swift-Hohenberg Equation.

Repeat the arguments of my lecture to derive the amplitude equation for the generalized Swift-Hohenberg equation

$$\partial_t u = ru - \left(1 + \partial_x^2\right)u + (\partial_x u)^2\,\partial_x^2 u \qquad \text{for a scalar real field } u(x,t)\,.$$

3. The Turing Instability.

In a visionary and extremely influential paper[1] Turing (1952) laid down basics principles of the theory of Morphogenesis. It arises due to the linear instability of reaction and diffusion of chemical species in a fertilized cell. We focus here on the case with only one spatial dimension and two chemical species characterized by the concentrations $\vec{u} = (u_1, u_2)$,

$$\partial_t \vec{u}(x,t) = \vec{f}(\vec{u}(x,t)) + \mathsf{D}\vec{u}(x,t) \qquad \text{with the diffusion matrix } \mathsf{D} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

and a nonlinear function $\vec{f}(\vec{u})$ representing the chemical reactions. Turing assumes that the chemical reactions admit a spatially uniform fixed point $\vec{u}_0$, where $\vec{f}(\vec{u}_0) = \vec{0}$. To analyze its linear stability we write $\vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1$.

(a) Show that to linear order in ε we then obtain the following equations of motion for $\vec{u}_1$

$$\partial_t \vec{u}_1(x,t) = \mathsf{A}\vec{u}_1(x,t)) + \mathsf{D}\vec{u}_1(x,t)\,.$$

How is A related to $\vec{f}$?

(b) Argue that both components of $\vec{u}_1$ must have the same x dependence $\propto \mathrm{e}^{\mathrm{i}kx}$ in order to obtain a consistent solution, and that the solution of the linear stability problem thus takes the form Hence,

$$\vec{u}_1 \propto \mathrm{e}^{\mathrm{i}kx} \left(\mathrm{e}^{\gamma_+ t}\, \vec{e}_+ + \mathrm{e}^{\gamma_- t}\, \vec{e}_- \right)$$

where $\gamma_\pm$ and $\vec{e}_\pm$ are the eigenvalues and eigenvectors of

$$\left[\mathsf{A} - k^2 \mathsf{D}\right] \vec{e}_\pm = \gamma_\pm \vec{e}_\pm\,.$$

(c) Discuss the linear stability in terms of $\det\left[\mathsf{A} - k^2\mathsf{D}\right]$ and $\mathrm{Tr}\left[\mathsf{A} - k^2\mathsf{D}\right]$. Which types of instabilities do arise?

[1] A.M. Turing: The Chemical Basis of Morphogenesis, Phil. Trans. R. Soc. Lond. B **237** (1952) 37–72. In Leipzig we have access to the reprint in the Bltn. Mathcal. Biology **52** (1990) 153. I strongly recommend to read the original paper. It is an exceptionally beautiful layout of ideas, and truly amazing because it was written well before microbiology was established.