Universität Leipzig
Fakultät für Physik und Geowissenschaften
Jürgen Vollmer

Sommersemester 2019
Fr, 15. 6. 2019

Pattern Formation and Nonlinear Dynamics

11. Patterns

1. The Onset of Thermal Convection.

Thermal convection arises when the Rayleigh number Ra surpasses a critical value Ra_c,

$$\mathrm{Ra} > \mathrm{Ra}_c = \frac{\alpha\, g\, d^3\, \Delta T}{\nu \kappa} .$$

The instability is promoted by the isobaric coefficient of thermal expansion α, the gravitational acceleration g , the depth of the layer d, and the temperature difference across the layer ΔT. It is suppresses by the kinematic viscosity ν, and the thermal diffusivity κ. In the table I give some values for room temperature and atmospheric pressure.

Fluid	α [K^{-1}]	ν [m^2/s]	κ [m^2/s]
Air	3×10^{-3}	2×10^{-5}	2×10^{-5}
Mercury	2×10^{-4}	1×10^{-7}	3×10^{-6}
Water	2×10^{-4}	1×10^{-6}	2×10^{-7}

(a) Why and how would the different physical properties promote or suppress the onset of thermal convection?

(b) Two identical systems of depth h are filled with air and mercury, respectively. We slowly increase the temperature difference ΔT for temperatures close to room temperature and at normal atmospheric pressure. Using the values provided in the table, determine in which fluid convection will be observed first.

(c) We consider a convection cell with lateral dimensions very large as compared to its depth d. In the middle of the system we add a bump to the bottom plate that has a height of a few percent of the layer depth, and lateral dimensions comparable to the depth d. How will it impact the onset of convection? What do you expect will happen to the first unstable patterns?

(d) How will the findings change for a ditch rather than a bump?

2. Linear Stability for Other Cubic Nonlinearities.

In the lecture we discussed the linear (in-)stability of the Swift-Hohenberg-equation. Go through the same discussion in order to explore what changes when its cubic nonlinearity u^3 is replaced with $u\,(\partial_x u)^2$,

$$\partial_t u(x,t) = r\,u(x,t) - u(x,t)\,\big(\partial_x u(x,t)\big)^2 - \big(1+\partial_x^2\big)^2\,u(x,t)\,.$$

3. Linear Stability of a One-Variable Reaction-Diffusion System.

Perform a linear-stability analysis for the one-variable on-dimensional reaction-diffusion system

$$\partial_t u(x,t) = f(u(x,t)) + D\,\partial_x^2 u(x,t)\,,$$

where $f : \mathbb{R} \to \mathbb{R}$ is *some* nonlinear function of u.

(a) Argue that $f(u_0) = 0$ for $u_0 = 0$. Why is this necessary for a reaction diffusion system?

(b) Which type of linear instability does the model show?

(c) What changes in more than one dimension, i.e. for

$$\partial_t u(\vec{x},t) = f(u(\vec{x},t)) + D\,\nabla^2 u(\vec{x},t)\,,$$

4. Linear Stability Analysis of a Coupled Map Lattice.

A coupled map lattice (CML) is a discrete-time, discrete-space dynamical system with a dynamical system running in each lattice point, which is perturbed by the systems in its surroundings. Here we consider the system

$$u_i^{t+1} = f(u_i^t) + D\,\big(f(u_{i-1}^t - 2\,f(u_i^t) + f(u_{i+1}^t)\big)\,,\quad \text{where} \quad i\in\mathbb{Z} \quad \text{and} \quad f(u) = \mu\,u\,(1-u)$$

is a logistic map with parameter μ. The state of the system at a given time t comprises the infinite set of numbers $(u_i^t, i \in \mathbb{Z})$. The coupling parameter D governs the interaction of neighboring states.

(a) Show that the uniform solution $(u_i^t = u_*, i \in \mathbb{Z})$ is a fixed point of the dynamics iff it is a fixed point of the logistic map, $f(u_*) = u_*$.

(b) Discuss the stability of such a fixed point as function of $\mu \in [0,4]$ and $D \in \mathbb{R}$.

(c) How does the most unstable mode look like for negative D?

(d) Determine the values $D(\mu)$ where the values u_i^t remain confined in the unit interval, when the initial condition is chosen to obey this condition.

(e) Adapt the classification scheme for the type of linear instabilities to address the linear stability of CMLs.