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Pattern Formation and Nonlinear Dynamics

9. Hamiltonian Chaos

1. Arnold's Cat Map (identical to 7.3, which we did not fully discuss, yet.

In 1967 Arnol'd and Avez published a book Ergodic Problems of Classical Mechanics where they demonstrate the action of an area-preserving Hamiltonian flow by the Cat Map,

$$\mathcal{C} : [0,1] \times [0,1] \to [0,1] \times [0,1] \quad \text{with} \quad \mathcal{C} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1$$

(a) Determine the eigenvalues $\lambda_\pm$ of the matrix and show that $\lambda_+ \lambda_- = 1$. With suitable scaling the variables x and y may hence represent angles in some Poincaré-map of a Hamiltonian dynamics.

(b) Disregard the modulo operation for a moment, and sketch the image of the unit square in $\mathbb{R}^2$ under the matrix. The modulo operation will have four branches, that drop the following integer parts of the coordinates (x, y): branch 1: $(0,0)$, branch 2: $(0,1)$, branch 3: $(1,1)$, branch 4: $(1,2)$. To which areas in $[0,1] \times [0,1]$ will the corresponding regions be mapped by $\mathcal{C}$? Where are the preimages of the areas?

(c) The intersection of images and preimages defines 16 regions in the unit square. Sketch the regions and mark them by symbols $(a.b)$ that indicate which branch was used in the previous iteration of the map (a), and which branch will be used next (b).

(d) Demonstrate that 1. The regions (1.1), (1.2), (2.1) and (2.2) form two disjoint areas that are mapped to four areas by $\mathcal{C}$. However, the area 21. $= \mathcal{C}(2.1)$ fully lies in the upper right half of the unit square. Therefore, there are no chaotic orbits that only involve the symbols 1 and 2. 2. The regions (2.2), (2.3), (3.2) and (3.3) form two disjoint areas that are mapped to four areas by $\mathcal{C}$. However, the area 332. is fully contained in region 4, and 223. lies fully in 1. Therefore there are no chaotic orbits that only involve the symbols 2 and 3.
Due to the symmetry of the map every chaotic orbit must have at least three symbols.

(e) Due to the modulo transformation the corners of the unit square give different views on the same fixed point. Show that the stable manifold of the fixed point lies in area 3, and that the unstable manifold lies in area 1 (with appropriate choices of images and preimages!). Find the intersection point of the manifolds, and discuss the dynamics of the associated homoclinic trajectory.

2. Hénon-Heiles model

In 1964 Hénon and Heiles suggested a model for the dynamics of galactic clusters It involves two coordinates $\vec{q} = (q_1, q_2)$ and the conjugated momenta $\vec{p} = (p_1, p_2)$ leads to the following Hamilton function

$$H(\vec{q}, \vec{p}) = \frac{1}{2}\left(\vec{p}^2 + \vec{q}^2\right) + q_1^2\, q_2 - \frac{q_2^3}{3}\,. \qquad (1)$$

Energy is conserved because the Hamiltonian does not explicitly depend on time, and one can show that there is not other conservation law.

(a) Determine the equations of motion.

(b) We adopt the energy E as a bifurcation parameter and consider the Poincaré section for $q_1 = 0$ and $p_1 > 0$. This gives rise to a Poincaré map

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix}_n \to \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}_{n+1} .$$

There are many implementations of this map available in the internet. Find one of them (or write your own), and explore how the map changes upon increasing E. How does it look like for $E < 1/12$? What changes when increasing E beyond $E_c = 1/12$?

(c) You will find that the points in the Poincaré section will always be confined to the interior of an egg-shaped region in the q_2-p_2-plane. Show that the boundary of the region is associated to the condition $q_1 = p_1 = 0$, and determine the curve.

(d) The Toda-Bruna criterion states that Hamiltonian dynamics can be chaotic only if the Hesse matrix $-\partial^2 U/\partial q_1 \partial q_2$ of the potential U has at least one positive eigenvalue. Explore what the criterion implies about the evolution of two nearby trajectories that obey the equations of motion $\ddot{q}_k = -\partial U/\partial q_k$. Why is the criterion necessary? Is it also sufficient?

(e) Evaluate the Toda-Bruna criterion for the Hénon-Heiles model, and show that it implies the critical value $E_c = 1/12$.

3. Driven Pendulum revisited.

In the last tutorium we discussed the Poincaré map for the driven pendulum. Consider values far away from the turning point, but still with sufficiently high energy that the energy dependence of the period is noticeable.

(a) Determine the piecewise linear map which describes the deformation of phase-space volume close to a phase-space point that returns to its position after n periods of the driving frequency.

(b) Identify the situations where the eigenvalues of the map are complex conjugated, and where there are two real eigenvaules.

(c) Compare your findings to numerical simulations on the driven pendulum. My Sage script for the driven pendulum is provide on the web page.