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Pattern Formation and Nonlinear Dynamics

8. Hamiltonian Dynamics

1. Reflecting balls.

We consider the reflection of a ball from the ground, the lower side of a table, and back. The ball is considered to be a sphere with radius R, mass m, and moments of inertia $m\alpha R^2$ (by symmetry they all agree). Its velocity at time t_0 will be denoted as $\dot{\vec{z}}_0$. It has no spin initially. $\vec{\omega}_0 = \vec{0}$. The velocity and the spin after the n^{th} collision will be denoted as $\dot{\vec{z}}_n$ and $\vec{\omega}_n$. We will disregard gravity such that the ball travels on a straight path in between collisions.

(a) Sketch the setup, and the parameters adopted for the first collision: The positive x axis will be parallel to the floor and the origin will be put into the location of the collision. Its direction will be chosen such that the ball moves in the x-z plane. Take note of all quantities needed to discuss the angular momentum with respect to the origin.

(b) Upon collision there is a force normal to the floor, $\vec{F}_\perp$, and a force tangential to the floor, $\vec{F}_\parallel$. The spin of the ball will *only* change due to the tangential force. The normal force $\vec{F}_\perp$ acts in the same way as for point particles (i.e., as discussed in our derivation of the Sinai billiard): The velocity in vertical direction reverses direction and preserved its modulus. Denote the velocity component in horizontal direction as $v_n = \hat{x} \cdot \dot{\vec{z}}$, and demonstrate that conservation of energy and angular momentum imply that

$$v_n^2 + \alpha R^2 \omega_n^2 = v_{n+1}^2 + \alpha R^2 \omega_{n+1}^2$$

$$v_n - \alpha R\, \omega_n = v_{n+1} - \alpha R\, \omega_{n+1}\,.$$

Show that the tangential velocity component will therefore also reverse its direction and preserves the modulus,

$$v_n + R\, \omega_n = -(v_{n+1} + R\, \omega_{n+1})\,.$$

(d) Determine $v_1(v_0, \omega_0)$ and $\omega_1(v_0, \omega_0)$ for the initial conditions specified above. Now, we determine $v_2(v_1, \omega_1)$ and $\omega_2(v_1, \omega_1)$ by shifting the origin of the coordinate systems to the point where the next collision will arise, and we

rotate by π to account for the fact that we collide at the lower side of the table. What does this imply for v_1 and ω_1? Continue the iteration, and plot v_1, v_2 and v_3 as function of α. Discuss the result for a sphere with uniform mass distribution (what does this imply for ω?), and a sphere with $\omega = 1/3$.
Hint: For the plot one conveniently implements the recursion, rather than explicitly calculating v_3.

(e) What changes in this discussion when the ball has a spin initially?

2. Driven Pendulum.

We consider a mathematical pendulum where the pivot is subjected to a time dependent oscillation $z(t)$. When θ being the angle formed by the orientation of the pendulum arm and gravity, then the mass of the pendulum will be at the (Cartesian) coordinates

$$\vec{q} = (L \sin\theta, z(t) - L \cos\theta)$$

(a) Determine the equations of motion of the pendulum, and show that the impact of $z(t)$ can be interpreted as a periodic modulation of the gravitational acceleration g.

(b) We will consider the situation where g alternates between two values $g + \delta$ and $g - \delta$ that each act over half a period T of the driving. There are three interesting limiting cases.

(c) The period of the driving is slow as compared to the eigenfrequency of the pendulum. Sketch a phase-space plot of the mathematical pendulum, and discuss the impact of the change of gravity on the trajectories.

(d) The period of the driving is fast as compared to the eigenfrequency of the pendulum. In that case the pendulum does not move very far in one period of the oscillation. For a given position in phase space $(\theta, \dot{\theta})$: How does the pendulum move when stitching the two pieces of the motion in each of the half period together? How can this be taken into account in terms of an effective position dependent but constant force? Take a particularly careful look at the point where the pendulum is standing upright.

*(d) The ratio of the driving frequency and of the eigenfrequency is a rational number n/m with small integers n and m. This case is called the resonant case. I will say more about it in my lecture next week.