Universität Leipzig
Fakultät für Physik und Geowissenschaften
Jürgen Vollmer

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Pattern Formation and Nonlinear Dynamics

7. Attractors, Saddles, and Fractal Sets in 2D Dynamics

1. The Baker's Transformation.

In 1937 Eberhard Hopf published a small book, Ergodentheorie that is considered to be the first mathematically sound *and* readable account of this topic. For illustration purposes he introduced the baker's map

$$\mathcal{B} : [0,1] \times [0,1] \to [0,1] \times [0,1] \quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \begin{pmatrix} \eta\, x \\ y/\eta \end{pmatrix} & \text{for} \quad 0 \le x < \frac{1}{2}\,, \\ \begin{pmatrix} \eta\, x - 1 \\ (y+1)/\eta \end{pmatrix} & \text{for} \quad \frac{1}{2} < x \le 1\,, \end{cases}$$

where η is a real number larger than two.

(a) Determine the fixed points of the map and their stability, and sketch the action of the map.

(b) Determine the Lyapunov exponents.[1]

(c) Work out the symbolic dynamics for the baker's map. What is the relation between coordinates (x, y) and symbol sequences $\{0,1\}^{\mathbb{Z}}$?

(d) Choose your own birth date of that of another person that you like. Consider the binary representation of the year as the forward part of the symbol sequence, and the binary representation of the four digits mmdd as backward part. Where is the region localized that contains initial conditions that share this part of their symbol sequence? How does the region evolve upon iteration by the baker's map.

(e) Show that the inverse of the baker's map is generated by the reflection at the main diagonal of the coordinate system

$$\mathcal{B}^{-1} = \mathcal{S} \circ \mathcal{B} \circ \mathcal{S} \qquad \text{with} \quad \mathcal{S} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

How is the special case $\eta = 2$ useful to discuss ergodic theory?

[1]Remark: The first Lyapunov exponent amounts to the growth rate of the distance of two neighboring points that are infinitesimally apart. The sum of the first two Lyapunov exponents describes the growth of an area spanned by three points. The sum of the first n Lyapunov exponents describes the growth of a hyper-volume spanned by $n + 1$ points.

2. Metastable States of a Harmonic Chain in a Periodic External Potential.

In Phys. Rev. B 30, 917 (1984) and Phys. Rev. B 32, 5731 (1985) Reichert and Schilling pointed out that amorphicity can be interpreted as spatial chaos. In this model the distances between neighboring particles are interpreted as a time series that is generated by a dynamical system that is defined in terms of a two-dimensional piecewise-linear map. Up to a linear coordinate transformation their map corresponds to the area preserving version of

$$\mathcal{M} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \begin{pmatrix} \alpha\, x \\ \beta\, y \end{pmatrix} & \text{for} \quad y \le y_0 + \frac{\alpha}{\beta}\, \frac{\beta-1}{\alpha-1}\, x\,, \\ \begin{pmatrix} \alpha\, x + 1 - \alpha \\ \beta\, y + 1 - \beta \end{pmatrix} & \text{else}\ . \end{cases}$$

(a) Under which conditions is the map area preserving? What does this imply for the slope of the line separating the two branches of the map?

(b) For some parameter values the inverse of $\mathcal{M}$ is again generated by the reflection $\mathcal{S}$ at the main diagonal of the coordinate system, analogously to the baker's transformation. What is required for this to hold? What does this condition imply for the symbolic dynamics?

(c) Let us focus on the area preserving case with $\beta > 3$. Show that the system can have a complete symbolic dynamics with a dynamics that agrees with the baker map on its saddle. What does this imply for the values of y_0?

(d) Consider the case $\beta = 4$, and explore what happens to the invariant set upon increasing y_0.
Instructions: The line $L(x) = y_0 + \alpha\,(\beta - 1)\, x/[\beta\,(\alpha - 1)]$ separates the branches of the map. At the point $(0, 1 - \alpha)$ it will first intersect the fractal containing all points of the invariant set. Zoom into this region and show that for each level of resolution of the fractal the line L will swipe over the box in that corner, and than there will be a parameter interval where it passes through gaps of the fractal. What does that imply for the decay of the number of valid symbol sequences as function of $1 - \alpha - y_0$?

3. Arnold's Cat Map.

In 1967 Arnol'd and Avez published a book Ergodic Problems of Classical Mechanics where they demonstrate the action of an area-preserving Hamiltonian flow by the Cat Map,

$$\mathcal{C} : [0,1]\times[0,1] \to [0,1]\times[0,1] \quad \text{with} \quad \mathcal{C} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1$$

(a) Determine the eigenvalues $\lambda_\pm$ of the matrix and show that $\lambda_+ \lambda_- = 1$. With suitable scaling the variables x and y may hence represent angles in some Poincaré-map of a Hamiltonian dynamics (we will soon discuss this in the lecture).

(b) Disregard the modulo operation for a moment, and sketch the image of the unit square in $\mathbb{R}^2$ under the matrix. The modulo operation will have four branches, that drop the following integer parts of the coordinates (x, y): branch 1: $(0, 0)$, branch 2: $(0, 1)$, branch 3: $(1, 1)$, branch 4: $(1, 2)$. To which areas in $[0, 1] \times [0, 1]$ will the corresponding regions be mapped by $\mathcal{C}$? Where are the preimages of the areas?

(c) The intersection of images and preimages defines 16 regions in the unit square. Sketch the regions and mark them by symbols $(a.b)$ that indicate which branch was used in the previous iteration of the map (a), and which branch will be used next (b).

(d) Demonstrate that 1. The regions (1.1), (1.2), (2.1) and (2.2) form two disjoint areas that are mapped to four areas by $\mathcal{C}$. However, the area $21. = \mathcal{C}(2.1)$ fully lies in the upper right half of the unit square. Therefore, there are no chaotic orbits that only involve the symbols 1 and 2. 2. The regions (2.2), (2.3), (3.2) and (3.3) form two disjoint areas that are mapped to four areas by $\mathcal{C}$. However, the area 332. is fully contained in region 4, and 223. lies fully in 1. Therefore there are no chaotic orbits that only involve the symbols 2 and 3.
Due to the symmetry of the map every chaotic orbit must have at least three symbols.

(e) Due to the modulo transformation the corners of the unit square give different views on the same fixed point. Show that the stable manifold of the fixed point lies in area 3, and that the unstable manifold lies in area 1 (with appropriate choices of images and preimages!). Find the intersection point of the manifolds, and discuss the dynamics of the associated homoclinic trajectory.