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# Pattern Formation and Nonlinear Dynamics

## 6. Two-Dimensional Maps

**1. The Henon Map.**

We explore the chaos in the Henon Map

$$\mathcal{H} : \begin{pmatrix} x_n \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a\, x_n^2 + y_n \\ b\, x_n \end{pmatrix}$$

(a) Show that $\mathcal{H}$ can be written as composition of three maps, $\mathcal{H} = \mathcal{R} \circ \mathcal{C} \circ \mathcal{B}$, where

$$\mathcal{B} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 1 + y - a\, x^2 \end{pmatrix} , \qquad \mathcal{C} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} b\, x \\ y \end{pmatrix} , \qquad \mathcal{R} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

What is the geometrical interpretation of these transformations?

(b) Determine the linear map that describes how small perturbations of $(x, y)$ grow in time. What are its eigenvalues in the present case? Calculate the product of the eigenvalues. What does it mean in terms of the evolution of phase-space volumes? How is it related to the linearized map for a general 2d dynamical system?

(c) The Henon Map has two fixed points. Find the fixed points and discuss their stability.

(d) Plot a bifurcation diagram for the Hénon map: for a fixed value of $b$ it should show which values are taken by $x$, plotted vertically as function of $a$ (analogously to the diagram for the logistic map). Ideally there should be a slider that allows us to explore how the diagram changes for different choices of $b$.

(e) Plot the points for fixed parameter $a$ and $b$ also in the $(x, y)$ plane. This script will need a slider for $a$ and for $b$.

(f) Consider a list of points with small distances between neighboring points. The first point of the list is a fixed point of the dynamics. For different initial conditions there are two set of lines that can be mapped out upon repeated forward iteration. What does these lines map out? What do you find for repeated backward iteration?

(g) Choose the standard parameter values $a = 1.4$ and $b = 0.3$. Plot the points obtained in (e) together with the four sets of lines. Do you see the horse shoe? How does it change upon changing $a$ and $b$?

**2. Kicked Rotor.**

The configuration of a rotor amounts to the setup of a horizontal mathematical pendulum: A mass $m$ is fixed at a (approximately) massless bar around a pivot fixed at the origin of the $(x, y)$ plane. The angle with respect to the positive $x$-axis will be denoted as $\theta$.

(a) Sketch the setup.

(b) Determine the Hamilton function for the motion of rotor, when no external forces are acting. Determine the equations of motion for $\theta$ and its conjugated momentum $p$. What is the solution of these equations?

(c) The motion described in (b) obeys two conservation laws. What are the conserved quantities? Why is it clear from the beginning that these quantities are conserved?

(d) Now we subject the rotor to periodic force kicks in the $y$-direction. The kicks will be taken to be a time-dependent potential of the form

$$\Phi(\theta, t) = A \cos\theta \sum_{n\in\mathbb{Z}} \delta(t - nT)$$

Here, $A$ is the amplitude of the kicks, $T$ is the period of the driving and $\delta(t)$ is a Dirac delta function that can be approximated by

$$\delta(t) = \lim_{\epsilon\to 0} \delta_\epsilon(t) \qquad \text{mit} \quad \delta_\epsilon(t) = \begin{cases} \epsilon^{-1} & \text{for} \quad -\frac{\epsilon}{2} \le t \le \frac{\epsilon}{2} \\ 0 & \text{else} \end{cases}$$

Determine the Hamiltonian equations of motion.

(e) Determine the map that describes the position $\theta_n$ and the momentum $p_n$ directly after a the $n$th kick. Is the period a relevant parameter of the dynamics?

**Hint:** Write the dynamics relating the coordinates $(\theta_n, p_n)$ as a two-step process: 1. free propagation where the rotor evolves according to the solution obtained in (b). 2. an instantaneous kick.

(f) Write a Script that follows the time evolution of a set of initial conditions $(\theta_0, p_0) \in [1, \epsilon] \times [0, \delta]$, i.e. a plot of $(\theta_n, p_n)$ with a slider for $n$. How does the map change upon increasing $A$? Do you see the horse shoe?

(g) How do your findings change upon introducing a small dissipation in the kicks? For instance the kinetic energy of the rotor can be decreased by a factor $\gamma$ with $(1 - \gamma) \ll 1$.