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Pattern Formation and Nonlinear Dynamics

Blatt 5. One-Dimensional Dynamics

1. Singularity Lines as Bifurcations Skeleton for the Logistic Map.

We consider the logistic map $f(x, \mu) = \mu\, x\,(1-x)$. Its maximum at $x_c = 1/2$ is called the critical point of the map. The iterates of this point

$$g_0(\mu) = \frac{1}{2} \qquad \text{and} \qquad g_k(\mu) = f(g_{k-1}(\mu), \mu) \quad \text{for} \quad k \in \mathbb{N}$$

are called the skeleton of the bifurcation diagram.

(a) Take a look at the Sage script that I provided for the logistic map, or write a similar (and likely much faster) program that also plots the bifurcation diagram. Add the first skeleton lines $g_k(\mu)$ with $0 < k \leq 10$ to the plot, and zoom into the range $3.3 \leq \mu \leq 3.87$.

(b) What changes in the bifurcation diagram at intersections and tangencies of skeleton lines? Verify your conjecture by discussion the mathematical meaning of tangencies and intersections: How are they connected to bifurcations?

(c) Modify your script in order to plot the bifurcation diagram for other versions of the logistic map, e.g. $1 - a\,x^2$ or $-1 + b\,(1 - x^2)$. Are the findings robust?

2. Coordinates and Symbol Sequences.

In this exercise we explore how the initial condition can be found that will give rise to a prescribed symbol sequence, and we will use these expressions to determine bifurcation parameters.

(a) Consider the tent map $T(x) = a - 1 - \sigma\, a\, x$ with $x \in [-1, 1]$ and $\sigma = \text{sgn}(x)$. This expression can be inverted, finding $x = [a - 1 - T(x)]/(\sigma\, a)$. Repeat this procedure in order to express x by its second iterate $T^2(x)$, and use induction to show

$$x = (a-1) \sum_{k=1}^{\infty} (-a)^{-k} \prod_{l=0}^{k-1} \sigma_l \qquad \text{with} \quad \sigma_l = \text{sgn}(T^l(x))\,.$$

(b) For the logistic map least cluttered expressions are obtained for $L(x) = 1 - (bx)^2$. Show that this leads to the representation

$$x = \frac{\sigma_0}{b} \sqrt{1 - \frac{\sigma_1}{b} \sqrt{1 - \frac{\sigma_2}{b} \sqrt{1 - \frac{\sigma_3}{b} \sqrt{1 - \dots}}}} \qquad \text{with} \quad \sigma_k = \operatorname{sgn}(L^k(x))\,. \tag{1}$$

(c) A representation of coordinates in terms of its symbol sequence can be used to determine the parameter value b_c where a periodic orbit first appears as a superstable periodic orbit, i.e. where the critical point $x_c = 0$ is part of its iterates. Turn Eq. (1) into a dynamical system for b that has a fixed point for a given period-three orbit. Discuss the stability of the fixed point. Solve the equation explicitly, and consult your results of exercise 1 to verify that this value amounts to the position of a period three window in the bifurcation diagram.
Optional: Determine the bifurcation values for the period doubling bifurcations.

3. Bifurcation Analysis of a Map with two Critical Points.

The map

$$f(x) = a\,x^3 + (1-a)\,x$$

has two critical points at $x_c = \pm\sqrt{(a-1)/(3a)}$ such that a symbolic dynamics must involve three symbols $\{-1, 0, 1\}$ for the rising, decaying and subsequent rising branch of the map, respectively.

(a) Calculate the bifurcation diagram. What is similar to the logistic map?

(b) Check out which asymptotic sets can be reached by choosing different initial conditions. Is the system ergodic? What happens when it is not.

(c) Add the skeleton lines for the critical points. What do they tell you about the bifurcations?