Universität Leipzig
Fakultät für Physik und Geowissenschaften
Jürgen Vollmer

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Pattern Formation and Nonlinear Dynamics

Blatt 3. Conjugation and Symbolic Dynamics

1. Topological Conjugation.

We consider the tent map and its conjugations by two maps $k_1(x) = x^2$ und $k_2(x) = \log(2-x)/\log 2$.

(a) Which dynamics are obtained by conjugation with respect to $k_1(x)$, $k_1^{-1}(x)$, $k_2(x)$, and $k_2^{-1}(x)$?

(b) What are the related invariant densities?

2. Invariant density of the Logistic Map.

In the lecture we derived that the Logistic Map has an invariant density

$$\rho_*(x) = \frac{1}{\pi}\,\big(x\,(1-x)\big)^{-1/2}$$

(a) Verify that this is a stationary solution of the Frobenius-Perron eqution for this dynamics.

(b) Write a Sage program that follows the time evolutions of the following initial conditions

$$\rho_L(x,0) = \begin{cases} 4 & \text{for} \quad 0 < x < 1/4 \\ 0 & \text{else} \end{cases}$$

$$\rho_M(x,0) = \begin{cases} 2 & \text{for} \quad 1/4 < x < 3/4 \\ 0 & \text{else} \end{cases}$$

$$\rho_N(x,0) = 1 - \cos(2\pi\, x)$$

3. Symbolic Dynamics.

We consider the map $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} (\eta - 1) + \eta\, x & \text{for} \quad x < 0 \\ (1 - \eta) + \eta\, x & \text{for} \quad x \geq 0 \end{cases}$$

(a) Determine the set of points that do not leave the interval $[-1, 1]$ in N iterations. How does it look like for $\eta > 2$? How for $1 < \eta < 2$.

(b) Show that x can be expressed as

$$x = (\eta - 1) \sum_{k=1}^{\infty} \sigma_{k-1} \, \eta^{-k} \qquad \text{with} \qquad \sigma_k = \text{sign}(f^k(x))$$

(c) Show that the trajectories that share the same ν symbols $\{\sigma_k, k = 0 \ldots \nu - 1\}$ lie in an interval of length $\eta^{-\nu}$. What does this imply about the convergence of a symbol sequence towards the associated value x?

(d) Determine the total length of the intervals that contain all distinct initial symbol sequences of lenth ν. How does the total length scale with ν? What does this imply about the uniqueness of the decomposition?

(e) Find an example of a point $x_* \in (-1, 1)$ that for a given fixed η has at least two different representations in the form indicated in (b).

(f) Determine the symbols σ_k based on the iteration $f^k(x)$. What do you find? What does this imply about the symbol sequences adopted by the symbolic dynamics?