Universität Leipzig
Fakultät für Physik und Geowissenschaften
Jürgen Vollmer

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Pattern Formation and Nonlinear Dynamics

Blatt 2. Basics in Dynamical Systems

1. Dimensional analysis and equations of motion

Consider a ball of mass m that can move without friction along a tube of mass M and length L. The tube is free to swing around a pivot at one of its end. Its angle with respect to the horizontal axis will be denoted as θ, and the position of the ball inside the tube as x, with origin at the pivot. We consider the initial condition $\theta_0 = 0$, $x_0/L = 0$, and zero intial velocities.

(a) Let η be the fraction of the tube that the ball has traversed by the time the tube becomes vertical. Does η depend on L?
Hint: This problem should be solved by dimensional analysis.

(b) Determinme the equations of motion of $\theta(t)$ and $x(t)$ by the Lagrange formalism. After choosing appropriate length and time units, the solution depends on a single dimensionless parameter. What does this say about the value of η?

(c) Employ Sage to numerically solve the equations of motion. How do the trajectories look like? How does η depends of the dimensionless parameter?

2. Phase-space flows and Poincaré maps for linear dynamical systems

We consider the linear dynamical systems

$$\ddot{x}(t) + a\,\dot{x}(t) + b\,x(t) + c \qquad \text{with} \quad a, b, c \in \mathbb{R}\,.$$

and expore their solutions for differnt choices of a, b, and c?

(a) For our analysis the parameter c may always be taken to be zero! Why is this justified?

(b) Consider the case $b = c = 0$. What type of motion is this? How do the solutions $x(t)$, $\dot{x}(t)$ look like? Sketch the solutions in the phase space $x, \dot{x}$).

(c) Consider the case $a = c = 0$ and $b > 0$. What type of motion is this? How do the solutions $x(t)$, $\dot{x}(t)$ look like? Sketch the solutions in the phase space $x, \dot{x}$).

(d) Consider the case $a = c = 0$ and $b < 0$. What type of motion is this? How do the solutions $x(t)$, $\dot{x}(t)$ look like? Sketch the solutions in the phase space $x, \dot{x}$).

(e) Consider the case $c = 0$ and $b^2 > 4\,a$. What type of motion is this? How do the solutions $x(t)$, $\dot{x}(t)$ look like? Sketch the solutions in the phase space $x, \dot{x}$).

(f) Consider the case $c = 0$ and $b^2 = 4\,a$. What type of motion is this? How do the solutions $x(t)$, $\dot{x}(t)$ look like? Sketch the solutions in the phase space $x, \dot{x}$).

(g) Consider the case $c = 0$ and $b^2 > 4\,a$. What type of motion is this? How do the solutions $x(t)$, $\dot{x}(t)$ look like? Sketch the solutions in the phase space $x, \dot{x}$).

(h) In case (g) almost all trajectories intersect the positive x-axis infinitely often. In such a situation the dynamics can be characterized by the Poincaré map $f(x)$ that provides the value of the $(n+1)^{\text{st}}$ intersection as function of the n^{th} intersection, i.e., $f x_{n+1} = (x_n)$. How does the Poincaré map look in this case?

3. Heat diffusion and convection

We consider a layer of a fluid with a linear temperature profile $T(z) = T_0 - G\,z$ in vertical direction z. It is maintained by heat diffusion with diffusion coefficient κ between the top and the bottom of the fluid that are maintained at temperatures T_0 at the bottom $z = 0$, and $T_h = T_0 - G\,h$ at the top $z = h$.

In this exercise we explore whether the heat transport in unstable with respect to convection. To this end we consider a blob that rises because it is slightly warmer than its environment. The sytem is unstable when the blob rises sufficiently fast to maintain the temperature difference. It is stable when the blob is slowed down by viscosity, such that the temperature fluctuattion decays.

(a) Let the blob radius be a, the temperature difference between blob and outside be δT, and the vertical blob velocity be U.
Estimate δT for a given temperature gradient G and heat diffusion κ.

(b) The buoyancy force $\mathcal{F}_{\mathcal{B}}$, that pushes the droplet upward, is due to gravitational accelleration $9.81\,\mathrm{m/s^2}$ and the thermal expansion of the fluid, $\delta\rho = -\alpha\,\rho_0\,\delta T$. Here, ρ_0 is the density of the surrounding fluid and $\rho_0 + \delta\rho$ the density of the blob.
How does $\mathcal{F}_{\mathcal{B}}$ depend on δT, ρ_0, g and α?

(c) The buoyancy force is counteracted by the drag force $\mathcal{F}_{\mathcal{D}}$, which is a function of the blob radius a, the rising velocity U, and the viscosity of the fluid $\eta = \nu\rho_0$. In the latter expression the kinematic viscosity ν is a diffusion coefficient that describes length- and time-scales for changes of the momentum of the fluid.
How does $\mathcal{F}_{\mathcal{D}}$ depend on ν, ρ_0, a and U?

(d) The Rayleigh number is obtained by substituting the blob radius by the layer height in the expressions for the forces, such that

$$\mathrm{Ra} = \frac{\mathcal{F}_{\mathcal{B}}}{\mathcal{F}_{\mathcal{D}}} = \frac{g\,\alpha\,G\,h^4}{\kappa\nu}$$

Convection arises when this dimensionless parameter exceeds a value of about 10^3. What does this mean in terms of the blob size?
For a pot of water with depth $h = 10\,\mathrm{cm}$ and room temperature $20\,°\mathrm{C}$ the instability arises roughly for a lower temperature of $50\,°\mathrm{C}$. Porridges and risottos have roughly the same density and thermal properties as water, while their kinematic viscosity is easily four order of magnitude larger than that of water. What does that imply for cooking?