Stochastic Processes — Tutorial 12

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Universität Leipzig
Institut für Theoretische Physik
Jürgen Vollmer

Exercise 1 *Revisiting the complex oscillator with noisy frequency*

In the lecture we discussed the complex oscillator

$$\mathrm{d}z(t) = (\mathrm{i}\,\omega - \gamma)\; z(t)\,\mathrm{d}t + \mathrm{i}\,\sqrt{2\gamma}\,z(t)\,\mathrm{d}W(t) \tag{1a}$$

where ω and γ are real constants, the noise $W(t)$ takes real values, and $z(t)$ is a function with complex values. It represents the motion of particles on a circle with fixed radius R_0, mean angular drift ω in angular direction φ, and noise that gives rise to a diffusivity γ.

(a) Consider a sharp initial distribution where all ensemble members start at $\varphi_0 = 0$. Show that the probability distribution $P(\varphi, t)$ for the angular coordinate takes then the form

$$P(\varphi, t) = \frac{N}{\sqrt{t}}\,\exp\left[-\frac{(\varphi - \omega t)^2}{2D\,t}\right] \quad \text{with } \varphi \in \mathbb{R}\,.$$

How are the normalization constant N and the diffusivity D related to γ?
How does the distribution look like for $\varphi \in [0, 2\pi]$?
Sketch the probability distribution for the initial value,
and values where $Dt \ll 2\pi$, $Dt \simeq 2\pi$, and $Dt \gg 2\pi$, respectively.

(b) The distribution is symmetric in the angle $\theta = \varphi - \omega\,t$. Use this symmetry to show that trajectories that the average distance $R(\theta)$ of trajectories arriving at θ and $-\theta$ will be proportional to $\cos\theta$. Proof that as a consequence the average of the positions $\vec{R}(t) = R_0(\cos\varphi(t), \sin\varphi(t))$ takes the value

$$\langle \vec{R}(t)\rangle = R_0\,\mathrm{e}^{-c\gamma t}\,(\cos(\omega t), \sin(\omega t))$$

where the exponential decay arises from

$$\mathrm{e}^{-c\gamma t} \propto \int \mathrm{d}\theta\,\cos\theta\; P(\theta, t)\,.$$

Determine c.

(c) Compare the results to the expectation value $\langle z(t)\rangle$ based on the expression derived in the lecture.

Exercise 2 *Ornstein-Uhlenbeck processes*

Processes described by stochastic differential equations of the form

$$\mathrm{d}x = -kx\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W(t) \tag{2}$$

are denoted as Ornstein-Uhlenbeck processes when k and D are real constants and $W(t)$ is delta-correlated white noise.

(a) Find the expectation value $\langle x(t) \rangle$.

(b) Find the variance Var(x, t).

(c) Determine the time autocorrelation function $\big\langle \big(x(t) - \langle x(t)\rangle\big)\,\big(x(s) - \langle x(s)\rangle\big)\big\rangle$.

Exercise 3 *Rate functions in large deviation theory*

The theory of large deviations deals with the asymptotic scaling of random variables s_N that comprise a sum of N variables,

$$s_N = \frac{1}{N}\sum_{k=0}^{N} x_k$$

where x_k are indendent variables with a prescibed distribution $P(x_k = x)$. The theory states that s_N obeys a *large deviation principle* when the distribution can be described by a rate function $I(s)$,

$$P(s_N = s) \asymp \mathrm{e}^{-N\,I(s)}\,, \tag{3}$$

where $\asymp$ holds that $\ln P(x) = -N\,I(x) + \mathcal{O}(\ln N)$.

(a) Assume that the values x_k are Gaussian random variables,

$$P(x_k = x) = \frac{1}{\sqrt{2\pi\sigma^2}}\;\exp\left[\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \text{with } x, \mu, \sigma \in \mathbb{R}\,.$$

Determine the probability distribution $P(s_N = s)$.
Discuss the rate function $I(s)$ and the corrections to the scaling.
What happens when x_k and μ are vectors in $\mathbb{R}^d$?

(b) Assume that the values x_k are independently selected from the probability distribution,

$$P(x_k = x) = \frac{1}{\mu}\;\mathrm{e}^{-x/\mu} \qquad \text{with } x, \mu \in \mathbb{R}_+\,.$$

Determine the probability distribution $P(s_N = s)$.
Discuss the rate function $I(s)$ and the corrections to the scaling.
Hint: Write $P(s_N = s)$ as the marginal with the constraint $s = s_N = N^{-1}\sum_k x_k$ of the probability distribution $P(x_1, \dots, x_N)$ to obtain the values $(x_1, \dots, x_N)$.

Exercise 4 *Properties of rate functions*

The scaling, Eq. (3), implies a number of properties of the rate function that are worth noting:

(a) In the lecture we argued that $I(s)$ is convex. Let s_0 be its (unique) minimum.
Observe now that

$$\langle s \rangle = s_0 + \int \mathrm{d}s\, (s - s_0)\, \mathrm{e}^{-N\, I(s)}\,.$$

and let $I(s) = I_0 + I_2\,(s - s_0)^2 + \dots$ be the Taylor expansion of $I(s)$ around its minimum.
Show that

$$\langle (s - s_0) \rangle \propto N^{-1}\, \mathrm{e}^{-N\, I_0} \quad \text{and} \quad \langle (s - s_0)^2 \rangle \propto N^{-3/2}\, \mathrm{e}^{-N\, I_0}\,.$$

Show that this implies $s_0 = \langle s \rangle$ and $I(s_0) = 0$.

(b) The *Law of Large Numbers* asserts that s_N converges to $\langle x \rangle$ for $N \to \infty$.
How is this related to the minimum of the rate function?

(c) The *Central Limit Theorem* asserts that for $N \to \infty$ the distribution $P(s_N = s)$ converges to a Gaussian function with maximum at $\mu = \langle x \rangle$ and variance Var(s) = Var$(s) = \sigma^2/N$.
How is this related to the shape of $I(s)$ close to its minimum?
In which range of s-values does the central limit theorem apply?

Exercise 5 (optional) *Rate Functions and Statistical Mechanics*

Consider a system with N spins that can take the values $\sigma_k \in \pm 1$, $k = 1, \dots, N$. The spins are interacting with an external field B, but there are no interactions between the spins. Hence, the Hamiltonian of the system takes the form

$$H(\{\sigma_k\}_{k=1}^N) = B \cdot M \qquad \text{with magnetization } \; M = \sum_k \sigma_k\,.$$

To simiplify the notations we absorb B into the energy scale such that $B = 1$.

(a) Show that the microcanonical partition sum, i.e. the number of states with $H = M$, can be written in the large deviation from

$$\Omega(H) \asymp \mathrm{e}^{-N\, I_1(m=H/N)} \qquad \text{with } I(m) = \frac{1-m}{2} \ln \frac{1-m}{2} + \frac{1+m}{2} \ln \frac{1+m}{2}\,.$$

How is $I_1(m)$ related to the entropy of the spin system?

(b) Show that the canonical partition sum $Z(\beta)$ obeys a large deviation principle

$$Z(\beta) := \sum_{k=0}^{N} \Omega(H = 2k - N)\, \mathrm{e}^{-\beta\,(2k-N)} \asymp \mathrm{e}^{-N\, I_2(\beta)}$$

How is $I_2(\beta)$ related to the free energy?
How is $I_2(\beta)$ related to the cumulant-generating function of $P(H)$?
What is the relation between I_1 and I_2 according to large deviation theory?
How does this compare to the relation between free energy and entropy in statistical physics?