Stochastic Processes — Tutorial 11

21 June 2018 — Ito formalism

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Exercise 1 *Expectation values of exponential functions*

In the lecture we used that

$$\left\langle \mathrm{e}^{x(t)} \right\rangle = \exp\left[\frac{1}{2}\left\langle x^2(t)\right\rangle\right] \qquad \text{for} \quad \langle x(t)\rangle = 0\,. \tag{1a}$$

(a) Proof that this is true.

(b) How does this relation look like when $\langle x(t)\rangle \neq 0$?

Exercise 2 *Multiplicative linear white noise*

A stochastic variable $x(t) \in \mathbb{R}$ subjected to multiplicative linear white noise evolves according to

$$\mathrm{d}x(t) = c\,x(t)\,\mathrm{d}W(t)\,. \tag{2}$$

Henceforth, we assume that $c \in \mathbb{R}$ is constant.

(a) Determine the mean displacement $\langle x(t)\rangle$ and the variance of $x(t)$ when Eq. (2) is interpreted as an Ito equation.

(b) How does the answer look like when the differentials are interpreted as an Stratonovich equation?

(c) Determine the correlation function $\big\langle [x(t) - \langle x(t)\rangle]\,[x(s) - \langle x(s)\rangle]\big\rangle$ both in Ito and in Stratonovich calculus.
Verify that the correlation functions reduce to the respective variances when $|t-s|$ approaches zero.

Exercise 3 *Change of variables in Ito calculus*

In the lecture I provided Ito's formula for a change of variables from x to $F(x)$,

$$\mathrm{d}x = a\,\mathrm{d}t + b\,\mathrm{d}W(t) \quad \Rightarrow \quad \mathrm{d}F(x) = \left[a\,F'(x) + \frac{b^2}{2}\,F''(x)\right]\mathrm{d}t + b\,F'(x)\,\mathrm{d}W(t)\,. \tag{3}$$

How does it generalize to functions $F(x,t)$ that depend explicitly on t?

Exercise 4 *Conversion between Stratonovich and Ito stochastic differentials*

In the lecture we derived the correspondence

$$\text{Stratonovich} \qquad \mathrm{d}x = a\,\mathrm{d}t + b\,\mathrm{d}W(t) \tag{4a}$$

$$\text{Ito} \qquad \mathrm{d}x = \left[a + \frac{b}{2}\frac{\partial b}{\partial x}\right]\,\mathrm{d}t + b\,\mathrm{d}W(t)\,, \tag{4b}$$

where a and b are functions of $x(t)$ and t.

(a) Find the reverse transformation from Ito to Stratonovich stochastic differentials.

(b) Derive the generalization to the multidimensional case.

Exercise 5 *Complex oscillator with noisy frequency*

In the lecture we discussed the complex oscillator

$$\mathrm{d}z(t) = (\mathrm{i}\,\omega - \gamma)\;z(t)\,\mathrm{d}t + \mathrm{i}\,\sqrt{2\gamma}\,z(t)\,\mathrm{d}W(t) \tag{5a}$$

where ω and γ are real constants, the noise $W(t)$ takes real values, and $z(t)$ is a function with complex values.

(a) Show that the differential $\mathrm{d}R^2 = z^*\,\mathrm{d}z + z\,\mathrm{d}z^*$ does *not* depend on the noise $W(t)$. What does this imply for the evolution of $z(t)$? Does $R(t)$ represent the evolution of the distance of the origin, $z(t)\,z^*(t)$, of individual trajectories?
If not: What does it describe? What is the appropriate expression for $\mathrm{d}(z(t)\,z^*(t))$?

(b) Write $z(t)$ in terms of polar coordinates $z(t) = R(t)\,\mathrm{e}^{\mathrm{i}\phi(t)}$ and proof that

$$\mathrm{d}\phi = (\omega - \mathrm{i}\gamma)\;\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}W(t)\,.$$

(c) Determine the expectation value and the variance of $\phi(t)$. The expectation value will turn out to be complex! Why is this compatible with the representation $z(t) = R(t)\,\mathrm{e}^{\mathrm{i}\phi(t)}$?

(d) Calculate $\langle z(t)\rangle$ also for the representation $z(t) = R_0\,\mathrm{e}^{\mathrm{i}\varphi(t)}$ where $R_0 = R(t_0)$.

(e) Compare the two approaches and determine also the variance and the two-time autocorrelation function of $z(t)$.

(Bonus) Compare the results to those of exercise 10.3.