Stochastic Processes **Tutorial 10**

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Exercise 1 *Probability distribution for velocities*

We consider the Langevin equation for velocity of a Brownian particle in an external force field

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \gamma\,(d - v(t)) + \sigma\, f(t)\,. \tag{1a}$$

Here, d denotes an asymptotic drift velocity due to the external field, γ the viscous damping, σ is the noise amplitude, and the normalized noise $f(t)$ obeys the relations specified in the lecture.

(a) Integrate the differential equation (1a) in order to show that

$$v(t) = d + (v_0 - d)\,\mathrm{e}^{-\gamma t} + \sigma\,\mathrm{e}^{-\gamma t}\int_0^t \mathrm{d}s\,\mathrm{e}^{\gamma s}\, f(s)\,. \tag{1b}$$

(b) Evaluate the cumulants of $v(t)$.
Hint: Beyond second order they should vanish.

(c) Show that for large times $\gamma t \gg 1$ the cumulant generating function takes the form

$$\ln\left\langle \mathrm{e}^{\mathrm{i}kv}\right\rangle = \mathrm{i}\,dk - \frac{\sigma}{2\gamma}\,k^2\,. \tag{1c}$$

(d) What does Eq. (1c) imply for the Fourier transformation of the velocity probability distribution $P(v,t)$? Determine $P(v,t)$.

(e) How should γ, σ and the temperature T be related to obtain a Maxwell-Boltzmann distribution for the velocity distribution?

(f) Which steps in this derivation change when $v(t)$ is a vector in $\mathbb{R}^3$ rather than a scalar quantity?

Exercise 2 *Mean-square displacement*

The displacement $x(t)$ is related to the velocity $v(t)$ via the integral

$$x(t) = \int_0^t \mathrm{d}t\; v(t)\,. \tag{2}$$

Henceforth, we assume that $v(t)$ evolves as described in exercise 1.

(a) Determine the mean displacement, the mean-square displacement, and the second cumulant of the distribution of the displacement.

(b) Does it matter whether one adopts the Ito or the Stratonovich definition for the stochastic integrals?

Exercise 3 *Correlation functions*

We consider the Langevin equation

$$x(t) = \mathrm{i}\omega\, x(t) + \sigma\, f(t)\,. \tag{3}$$

(a) Which type of physics is described by this model?

(b) Integrate the model to show that

$$x(t) = x_0\, \mathrm{e}^{\mathrm{i}\omega t} + \sigma\, \mathrm{e}^{\mathrm{i}\omega t} \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\omega s}\, f(s)\,.$$

(c) Determine the expectation value $\langle x(t) \rangle$.

(d) Determine the correlation function

$$C(s,t) = \langle x(s)\, x^*(t) \rangle - \langle x(s) \rangle \, \langle x^*(t) \rangle \,.$$

Determine also the variance of the distribution of $x(t)$.

(e) Provide a physical interpretation for the evolution of the ensemble.