Stochastic Processes — Tutorial 8

31 May 2018 — Laplace transformations and first-passage times

Universität Leipzig
Institut für Theoretische Physik
Jürgen Vollmer

Exercise 1 *Laplace Transformations and Solutions of Initial Value Problems*

We consider the driven damped harmonic oscillator

$$\ddot{x}(t) + \gamma\,\dot{x}(t) + \omega_0^2\,x(t) = F(t) \tag{1}$$

with extenal forcing

$$F(t) = A\,\mathrm{e}^{\mathrm{i}\Omega t} \qquad \text{with} \quad A \in \mathbb{R}\,,$$

where $x(t) \in \mathbb{C}$ is the amplitude of the oscillator at time t.

(a) Show that Re$(x(t))$ is the solution of this differential equation with forcing Re$(F(t))$.

(b) Determine the Laplace transformation of the forcing

$$F(s) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-st}\,F(t)\,.$$

(c) Start from Eq. (1) to show that $x(s)$, the Laplace transformation of $x(t)$, takes the form

$$x(s) = \frac{F(s) + I(s)}{s^2 + \gamma\,s + \omega_0^2}\,.$$

where $I(s)$ is an expression involving the initial conditions $x_0 = x(0)$ and $v_0 = \dot{x}(0)$. Determine $I(s)$.

(d) Show that

$$x(-\mathrm{i}s) = \int_{\mathbb{R}} \mathrm{d}t\,\mathrm{e}^{\mathrm{i}st}\,\Theta(t)\,x(t)$$

where $\Theta(t)$ is the Heavyside function. It takes the value 1 for $t \geq 0$ and 0 for $t < 0$.

(e) Determine $\Theta(t)\,x(t)$ by taking the inverse Fourier transformation of $x(-\mathrm{i}s)$. We will only be interested in the long-time asymptotics. You may neglect all term that decay for large t.

Exercise 2 *Greens functions for biased diffusion*

We look for a solution of the diffusion equation

$$\partial_t \rho(\vec{x},t) = -\vec{v}\cdot\nabla\,\rho(\vec{x},t) + D\,\nabla^2\rho(\vec{x},t)\,, \quad \text{with} \quad t\in\mathbb{R}\,, \quad \vec{x}\in\mathbb{R}^d \tag{2a}$$

that takes the form

$$\rho(\vec{x},t) = \int_{\mathbb{R}} \mathrm{d}^d x_0\, G(\vec{x}-\vec{x}_0,t)\, I(\vec{x}_0) \tag{2b}$$

with

$$G(\vec{x},0) = \delta(\vec{x})\,, \tag{2c}$$

$$G(\vec{x},t) = 0 \qquad \text{for} \quad t<0\,. \tag{2d}$$

(a) Verify that $I(\vec{x})$ is the intial condition for the density, i.e., $\rho(\vec{x},0)=I(\vec{x})$.

(b) Verify that $G(\vec{x},t)$ is a solution of the differential equation

$$\mathcal{D}G(\vec{x},t) = 0$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D} = \partial_t + \vec{v}\cdot\nabla - D\,\nabla^2\,.$$

(c) Evaluate the Fourier transformation

$$0 = \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\, \mathcal{D}G(\vec{x},t)$$

in order to show that

$$G(\vec{k},\omega) = \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\, G(\vec{x},t) = \frac{G(k,t=0)}{D\,\vec{k}^2 - \mathrm{i}\vec{v}\cdot\vec{k} - \mathrm{i}\omega}\,.$$

Hint: How did the initial condition appear in the Laplace transform?

(d) Show that $G(k,t=0)=1$.

(e) Show that

$$G(\vec{k},t) = \frac{1}{2\pi}\int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t}\, G(\vec{k},\omega) = \begin{cases} 0 & \text{for} \quad t<0\,, \\ \exp\bigl(\mathrm{i}\vec{v}\cdot\vec{k}t - D\vec{k}^2 t\bigr) & \text{for} \quad t\geq 0\,. \end{cases}$$

(f) Determine the Greens function $G(\vec{x},t)$ for the diffusion operator $\mathcal{D}$.

Exercise 3 *First-passage times for a biased random walk*

In the class we calculated the discrete Fourier-Laplace transformation for a biased random walk on $\mathbb{Z}$ with jump probabilities r and l to the right and left, respectively. For $r + l + 1$ and the initial condition $P(x, 0) = \delta_{x,x_0}$ it takes the form

$$P(k, z) = \frac{\mathrm{e}^{\mathrm{i}kx_0}}{1 - u(k)\, z}\,, \quad \text{with} \quad u(k) = r\,\mathrm{e}^{\mathrm{i}k} + l\,\mathrm{e}^{-\mathrm{i}k}\,.$$

In this exercise we determine the Laplace transform

$$P(x, z) = \sum_{N=0}^{\infty} z^N\, P(x, N) \tag{3}$$

of the probability distribution $P(x, N)$, and we use it to calculate the distribution of first passage times.

(a) Throughout the exercise we assume that $l \leq r$. How would one find the solution for $r > l$?

(b) Rather than evaluating Eq. (3) for the binomial distribution $P(x, N)$, it is easier to determine $P(x, z)$ based on $P(k, z)$. Which integral should be performed to invert the discrete Fourier transformation? Show that an appropriate substitution $\rho(k)$ transforms this integral into

$$P(x, z) = \frac{\mathrm{i}}{2\pi} \oint_{\mathcal{U}} \mathrm{d}\rho\, \frac{\rho^{x-x_0}}{zl\,\rho^2 - \rho + zr}$$

where the integral contour $\mathcal{U}$ denotes the unit circle.

(c) Show that for $0 < z < 1$ the integral has a single root in the unit cirle,

$$\rho_c = \frac{1 - \sqrt{1 - 4rlz^2}}{2zl}\,.$$

(d) Perform the contour integration and show that the Laplace transformation of the first passage time takes the form

$$F(x, z) = \begin{cases} 1 - \sqrt{1 - 4rlz^2} & \text{for} \quad x = x_0\,, \\ \left(\dfrac{1 - \sqrt{1 - 4rlz^2}}{2zl}\right)^{x-x_0} & \text{for} \quad x \neq x_0\,. \end{cases}$$

(e) In the time discrete case the first-passage times can be found by a Taylor expansion of $F(x, z)$ around $z = 0$. Show that

$$F(x_0, N) = \begin{cases} 0 & \text{for} \quad N \text{ odd}, \\ \dfrac{2\,(N-2)!}{\frac{N}{2}!\,\left(\frac{N}{2} - 1\right)!}\, r^{N/2}\, l^{N/2} & \text{for} \quad N \text{ even}. \end{cases}$$

Why is $F(x_0, N) = 0$ for N odd? Evaluate the result for $N = 2, 4, 6$ and check it by explicitly listing the possible paths for the first return in N steps.
Hint: $\ln(1 + x) = 1 + 2\,\sum_{k=1}^{\infty} (2k-2)!\, x^k / [2^k\, k!\, (k-1)!]$.

(f) Discuss the first-passage times for $F(x_0 \pm 1, N)$.