Stochastic Processes **Tutorial 7**

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Exercise 1 *The classical Flucutation-Dissipation Theorem*

In the derivation of the Fluctuation Dissipation Relation we used

$$\langle B(t)A(0)\rangle = \langle A(0)\, B(t + \mathrm{i}\hbar\beta)\rangle\,.$$

(a) Proof that the relation holds.

(b) What is the order of magnitude of $\hbar\beta$? How does it compare to time scales that can be realized in experiments in classical macroscopic systems? Can you find an experimental setting where $\beta t/\hbar$ will be of order one?

(c) Assume that $B(t)$ is analytical in t such that it can be expanded in a Taylor expansion,

$$B(t + \mathrm{i}\hbar\beta) = \sum_{j=0}^{\infty} \frac{(\mathrm{i}\hbar\beta)^j}{j!}\, B^{(j)}(t)\,.$$

Use this expansion to evaluate the integral in

$$T_{BA}(\omega) - T_{AB}(-\omega) = \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \left\langle \left[B(t), A(0)\right] \right\rangle\,.$$

Compare your result to the small $\beta\hbar\omega$ limit of the expression given in the lecture.

Exercise 2

— will be inserted —

Exercise 3 *Generalized Fokker-Planck Equation for the Poisson process*

The Poisson process models the statistics of counts where discrete signals arrive at random times with a uniform rate ν. An example is the number of registered events encountered in an experiment at a synchroton beamline (e.g., the number of Higgs Bosons identified at CERN). The stocastic variable is the number of counts $N(t)$ after beam time t. Now we explore the distribution $P(N, t)$ of the variable $N(t)$.

(a) Why is it justified to model this problem as a Markov process? What is the set $\mathcal{S}$ of states? Which transmission are admissible and what are the rates t_k^j, $j, k \in \mathcal{S}$?

(b) Go back to the analysis of $\frac{\mathrm{d}}{\mathrm{d}t}\langle M(a,t)\rangle$ in the lecture. Since $\mathcal{S}$ is discrete the integrals turn in to sums. Verify that in this case the generalized diffusivities $D_n(j,t)$ take the form

$$D_n(j,t) = \frac{1}{n!} \sum_{k\in\mathcal{S}} (k-j)^n \, t^j_k \, .$$

Determine $D_n(j,t)$ for the Poisson process! Write down the generalized Fokker-Planck equation.
Hint: If everything went fine the generalized Fokker-Planck equation should be a Taylor expansion of the Master equation

$$\partial_t P(N,t) = \nu \left[P(N-1,t) - P(N,t) \right] .$$

(c) Demonstrate that the Master equation is solved by the Poisson distribution

$$P(N,t) = \frac{\big(\mu(t)\big)^N}{N!} \, \mathrm{e}^{-\mu(t)} \, ,$$

with an appropriately chosen function $\mu(t)$. Determine $\mu(t)$.

(d) Determine the expectation $\langle N(t)\rangle$ based on the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle N\rangle_t = \langle D_1\rangle_t \ ,$$

and compare the result to the expectation of the Poisson distribution.

(e) Determine the second moment $\langle N^2\rangle_t$ based on the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle N^2\right\rangle_t = 2 \left\langle D_1 \, N\right\rangle_t + 2 \left\langle D_2\right\rangle_t \ ,$$

and compare the resulting expression of the variance to the one of the Poisson distribution.

Exercise 4 *Burst Theorem*

The Burst theorem states that if any D_n for $n > 2$ exists then there is an infinite set of D_n. In the lecture I indicated that this can be seen based on the inequality[1]

$$D^2_{m+n} \leq D_{2m} \, D_{2n} \, , \quad \text{for } m,n > 0 \, . \tag{4a}$$

(a) Why do we need the condition $m,n > 0$?

(b) Show that when $D_{2m} = 0$ then $D_k = 0$ for all $k > m$.

(c) Show that when $D_m \neq 0$ then $D_k \neq 0$ for all positive k in $\{2 + 2^j\,(m-2), j\in\mathbb{N}_0\}$.

(d) Show that therefore either $D_m = 0$ for all $m > 2$ or they all differ from zero.

[1] A proof of the inequality is provided in the appendix.

Appendix. *Proof of the inequality Eq. (4a)*

Let

$$
\begin{aligned}
a &= \frac{1}{n!} \lim_{\Delta t \to 0} \frac{(a(t+\Delta t) - a(t))^n}{\Delta t} \\
b &= \frac{c}{n!} \lim_{\Delta t \to 0} \frac{(a(t+\Delta t) - a(t))^m}{\Delta t}
\end{aligned}
$$

with some fixed number $c \in \mathbb{R}$. Then we have

$$
D_{m+n} = c \, \langle a \; b \rangle \, , \qquad D_{2n} = \langle a^2 \rangle \, , \qquad D_{2m} = c^2 \, \langle b^2 \rangle
$$

such that Eq. (4a) is equivalent to

$$
\langle a \, b \rangle^2 \leq \langle a^2 \rangle \, \langle b^2 \rangle
$$

for some value of c that will be chosen appropriately.

To derive the latter equation we observe

$$
\begin{aligned}
0 \leq \langle (a \pm b)^2 \rangle \quad &\Leftrightarrow \quad 2 |\langle a \, b \rangle| \leq \langle a^2 \rangle + \langle b^2 \rangle \\
&\Leftrightarrow \quad 4 \, \langle a \, b \rangle^2 \leq 4 \, \langle a^2 \rangle \, \langle b^2 \rangle + \big(\langle a^2 \rangle - \langle b^2 \rangle \big)^2 \; .
\end{aligned}
$$

However, the second term on the right hand side vanishes when we choose $c = \sqrt{\langle a^2 \rangle / \langle b^2 \rangle}$.
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