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Exercise 1 *Noise spectrum for measured noise*

The relation between the fluctuations $\alpha(t)$ of an observable $\Omega(t)$, and the flucutations $\alpha_{\text{out}}(t)$ in the measured signal $\Omega_{\text{out}}(t)$ can be expressed through a filter, $K(t)$,

$$\alpha_{\text{out}}(t) = \int_{-\infty}^{t} K(t-s)\, \alpha(s)\, \mathrm{d}s\,.$$

(a) Causality implies that $K(t) = 0$ for $t < 0$. What does this imply for the integral

$$\int_{-\infty}^{\infty} K(t-s)\, \alpha(s)\, \mathrm{d}s\,?$$

(b) Let $\alpha(\omega)$ and $\alpha_{\text{out}}(\omega)$ be the Fourier transforms of $\alpha(t)$ and $\alpha_{\text{out}}(t)$. Show that

$$\alpha_{\text{out}}(\omega) = k(\omega)\ \alpha(\omega)\,.$$

Determine $k(\omega)$.

(c) Let $G(x)$, $x \in \{\alpha(t), \alpha_{\text{out}}(t)$ be the average noise intensity

$$G(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{t_0-T}^{t_0+T} |x|^2\, \mathrm{d}t\,.$$

Under which condition will $G(x)$ *not* depend on t_0? Show that

$$G(\alpha_{\text{out}}) = |k(\omega)|^2\ G(\alpha)\,.$$

(d) For an ideal measurement one would like to approach $G(\alpha_{\text{out}}) = G(\alpha)$ as closely as possible. What does this imply for the filter function $K(t)$?

Exercise 2 *Relatizing a filter in an electrical circuit*

Consider the voltage in an RLC circuit as signal, $U(t)$, and the voltage over the resistor as the output of a measurement signal, $U_{\text{out}}(t)$. We consider now the relation between the fluctuations $\alpha(t)$ in $U(t)$, and the flucutations $\alpha_{\text{out}}(t)$ in the measured signal $U_{\text{out}}(t)$.

(a) Show that

$$k(\omega) = \frac{R}{R + \mathrm{i}\,(\omega L - (\omega C)^{-1})}\,.$$

(b) Sketch the function $|k(\omega)|^2$.

(c) Show that the measurement will approach the ideal limit when R approaches zero. What will be the bandwidth of the filter? Which frequency will it pick out?

Exercise 3 *Fluctuation Relations*

We consider the observable $\sigma_k^j = \ln\bigl(r_k^j / r_j^k\bigr)$ for a Markov process with dynamically reversible transition rates r_k^j between the states j and k. Let $\tau(t)$ we a trajectory of this process, and $\Sigma(\tau, t)$ the value observed when σ_k^j is integrated along the trajectory.

(a) Show that the cumulant generating function, $Z(\vec{q})$, for the cumulants of the distribution of $\Sigma(\tau, t)$ obeys the symmetry

$$Z(\vec{q}) = Z(\vec{1} - \vec{q})$$

where $\vec{1}$ is the vector whose entries are all one.

(b) The proof is easier when one rather considers the observable

$$\omega_k^j = \ln \frac{p_j r_k^j}{p_k r_j^k} \,,$$

where p_j is the steady-state probability density of state p_j.
Where does this help? Why is it admissible?

(c) Verify that the process also fulfills the following fluctuation relation for the probability $P(\Sigma, t)$ to find the value Σ for $\Sigma(\tau, t)$:

$$\lim_{t \to \infty} \ln \frac{P(\Sigma, t)}{P(-\Sigma, t)} = \Sigma \,.$$

(d) Check that this fluctuation relation holds trivially for

- the displacement in a random walk on a line with probabilities r and l to take a step to the right and left, respectively. Steps are taken at integer times and $r + l < 1$.
- the displacement in a random walk on a line with rates r and l to take a step to the right and left, respectively.
- a Gaussian distribution.

(e) Provide a sketch of the distribution and provide a geometric interpretation of the fluctuation relation.