Stochastic Processes **Tutorial 3**

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Exercise 1 *Cycle Representation of Stationary States*

Proof the following statements about stationary states of Markov processes on a finite graph $\mathcal{G}$.

(a) Equilibrium states are stationary states.
Hint: Equilibrium states obey detailed balance $w_j^i \, p_i = w_i^j \, p_j$ for all pairs i, j of states.

(b) The time evolution of the probability p_k to find a realization of a Markov process in state k evolves according to

$$\dot{p}_k = \sum_{i, i \neq k} J_k^i \qquad \text{with} \quad J_k^i = w_k^i \, p_i - w_i^k \, p_k \, . \tag{1a}$$

(c) In a steady state the currents can be written as superpostions of cycle currents

$$J_k^i = \sum_{\alpha \in \mathsf{X}} J_\alpha \, \Theta_\alpha(i \to k) \qquad \text{with} \quad \Theta_\alpha(i \to k) = \begin{cases} 1 & \text{if} \quad i \to k \in \zeta_\alpha \\ -1 & \text{if} \quad k \to i \in \zeta_\alpha \\ 0 & \text{else} \end{cases} \tag{1b}$$

where X is a set of chords for the graph $\mathcal{G}$.
Hint: What happens when $i \to k$ is a chord? Use Kirchhoffs law to determine what happens on the other edges of the graph.

(d) The time derivative $\dot{p}_k$ vanishes when inserting Eq. (1b) into the right-hand side of Eq. (1a).

Exercise 2 *Coin Tossing*

We repeat Rosenkrantz experiment of coin flipping. Initially, we see head. We flip again when we see head. However, once we encounter tail we are happy, stop, and reply tail on any further inquiry. Let the coin flip be biased with a probability p to encounter head.

(a) Sketch the graph for this Markov process and mark the transfer probabilities.
Do we have dynamical reversibility? Do we have an absorbing state?

(b) Write down the transition matrix and determine its eigenvalues. In the lecture we showed that the entries of the columns of the transition matrix add to zero and that there is a zero eigenvalue. What is different here?

Determine the left and right eigenvectors $\langle 1|$ and $|1\rangle$ of the steady state, and the eigenvectors $\langle d|$ and $|d\rangle$ representing transients. In this notation d represents the probability of decay for the transient state.

(c) Show that the probability distribution $\vec{P}(N) = |N\rangle$ after N flips takes the form

$$|N\rangle = p^N \; |H\rangle + (1 - p^N) \; |T\rangle \qquad \text{with} \quad |H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; \text{and} \;\; |T\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Exercise 3 *Reaction with two decay channels*

Let I be a substance that decays into the stable substances A and B with a rates a and b, respectively.

(a) Sketch the graph for this Markov process and mark the transition rates.

(b) Write down the transition matrix. Find its eigenvalues, the eigenvectors $|A\rangle$, $|B\rangle$ and $\langle A|$, $\langle B|$ of the steady states, and the eigenvectors $|d\rangle$, $\langle d|$ representing transients that decay with a rate d.

(c) Represent the state of the system as

$$\vec{P}(t) = |t\rangle = \rho_A(t) \; |A\rangle + \rho_B(t) \; |B\rangle + \rho_I(t) \; |I\rangle \quad \text{with} \quad |I\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ,$$

and determine the time evolution for a sample that initially contains only substance I.

Exercise 4 *Three interacting particles on four sites*

We consider a system with four binding sites on a ring. In this system there is one $+$ particle and two $\circ$ particles. Accordingly, there is a tight state $T = [\circ, +, \circ, \_]$ where $+$ has two neighbors, as well as states $L = [\circ, +, \_, \circ]$ and $R = [\circ, \_, +, \circ]$ where $+$ only has a neighbor to the left or right, respectively. Let T decay to L and R with the same rate d, while the reverse processes happen at a rate u. Moreover, the hopping of $+$ is biased: $L \to R$ occurs with rate r and $R \to L$ with rate l.

(a) Sketch the graph for this Markov process and mark the transition rates.
Do we have dynamical reversibility? Do we have an absorbing state?

(b) Write down the transition matrix W and the skewed transition matrix Ŵ for an antisymmetric observable ω.
Remark: The (i, j) component of the transition matrix is the transition rate w^i_j to go from state $i \to j$. The (i, j) component of the skewed transition matrix is $w^i_j \, \mathrm{e}^{s\,\omega^i_j}$.

(c) Determine the characteristic polynomial of the skewed transition matrix,

$$P(\lambda; s) = a_3 \, \lambda^3 + a_2 \, \lambda^2 + a_1 \, \lambda + a_0$$

where (in principle) a_3, a_2, a_1, and a_0 are functions of s (see, however, (d)).

(d) Show that a_2 and a_1 do not depend on s, and that a_0 only depends on $\bar{\omega}\, s$, where $\bar{\omega}$ is the sum $\bar{\omega} = \omega_L^T + \omega_R^L + \omega_T^R$ over the contribution to the observable added up for the transitions around the cycle $T \to L \to R \to T$.

(e) Verify that $a_0(s)|_{s=0} = 0$. Why should that be true for every transition matrix?

(bonus) Consider the implicit equation

$$0 = P(\lambda(s); s)\,.$$

Use $\lambda(s)|_{s=0} = 0$ and the implicit function theorem to show that

$$J = \left.\frac{\mathrm{d}\lambda(s)}{\mathrm{d}s}\right|_{s=0} = \frac{-1}{a_1}\, \left.\frac{\mathrm{d}a_0(s)}{\mathrm{d}s}\right|_{s=0}$$

Do you see why this is true for *every* finite Markov process?

(f) Determine the current J for the present example.
How does it differ for different observables ω?