Stochastic Processes — Tutorial 2

19 April 2018 — Joint Probs and Stoch Processes

Universität Leipzig
Institut für Theoretische Physik
Jürgen Vollmer

Exercise 1 *Cumulants for the biased random walk*

We consider the random walk on $\mathbb{Z}$ where the walker is either taking a step to the left or to the right in each time step. The probabilies to go left and right will be denoted as l and r, respectively, where $l + r = 1$. For a realization of the random walk with N steps the displacement x_N will refer to the number of steps to the right minus the number of steps taken to the left.

(a) Verify by comparison to Exercise 1.4 that the characteristic function for x_N amounts to

$$\Phi_N(t) := \left\langle \mathrm{e}^{x_N \mathrm{i}t} \right\rangle = \left(r\,\mathrm{e}^{\mathrm{i}\,t} + l\,\mathrm{e}^{-\mathrm{i}\,t} \right)^N$$

(b) Determine the cumulant generating function.

(c) Determine the first three cumulants of this random walk.

(d) What do you observe for the third cumulant, as compared to a Gaussian distribution?

Exercise 2 *Marginal Probabilities for the 2d Gaussian Distribution*

We consider a Gaussian distribution in the two variables,

$$P(x, y) = \frac{1}{2\pi\sigma^2}\,\mathrm{e}^{-(x^2+y^2)/2\sigma^2} \qquad \text{with} \qquad (x, y) \in \mathbb{R}^2\,.$$

(a) Determnine the marginal probability densities $p_1(x) = \int \mathrm{d}y\, P(x, y)$ and $p_2(y) = \int \mathrm{d}x\, P(x, y)$.

(b) Demonstrate that the conditional probability density $P(x|y) := P(x, y)/p_2(y)$ amounts to $p_1(x)$. Is there a faster way to see that x and y are independent variables?

(c) Determine the probability density $p_3(R)$ to find a value (x, y) with modulus $R = \sqrt{x^2 + y^2}$.

(d) Determine the probability density $p_4(\phi)$ to find a value (x, y) in direction ϕ with respect to the positive x axis.

Exercise 3 *Uncorrelated vs. independent variables*

We consider the following probability distribution

$$p(x, y) = c\,(x^2 + y^2) \qquad \text{for} \qquad (x, y) \in [-1, 1] \times [-1, 1]\,.$$

(a) Determine the normalization constant.

(b) Determine the marginal probabilities $p_1(x) = \int \mathrm{d}y\, p(x,y)$ and $p_2(y) = \int \mathrm{d}x\, p(x,y)$.

(c) Determine the conditional probabilities $p(x|y)$ and $p(y|x)$.

(d) Determine the expectation values $\langle x \rangle$, $\langle y \rangle$, and $\langle x\, y \rangle$.

Exercise 4 *Stationary solutions for Markov processes*

A stochastic process is called *stationary* when its moments are not affected by a shift in time.

(a) Consider the biased random walk discussed in Exercise 2.1. Is this a stationary process?

(b) Now we consider a random walk on $\mathbb{Z}$ with position-dependent probabilities

$$r_j = K\, \mathrm{e}^{-\beta j} \quad \text{to go from site } j \text{ to } j+1,$$
$$l_j = K\, \mathrm{e}^{\beta j} \quad \text{to go from site } j \text{ to } j-1.$$

In these expressions β and K take constant values in $\mathbb{R}^+$ and $[0,1]$, respectively.

Let $X_t \in \mathbb{Z}$ be the position of a random walker at time t. Verify that the probability to find the walker at time t at lattice site $j \in \mathbb{Z}$ is a Markov process.

(c) This process admits an equilibrium state $p_j^{\text{eq}}, j \in \mathbb{Z}$ that obeys

$$r_j\, p_j^{\text{eq}} = l_{j+1}\, p_{j+1}^{\text{eq}} \qquad \text{for all} \quad j \in \mathbb{Z}\,.$$

Clearly, it defines a stationary Markov process.

(d) Determine the probability distribution $p_j^{\text{eq}}, j \in \mathbb{Z}$ and its cumulants.

Exercise 5 (bonus) *Bertrand's paradox*

A line is dropped randomly on a circle. The intersection will be a chord. What is the probability that the length of the chord is larger than that of a side of the inscribed equilateral triangle?

Bertrand's paradox states that the resulting probabiliy depends on the rule how the lines are selected. To understand this observation we consider intersections with the unit circle at the origin.

(a) Specify the line by two points: The first point is $P_1 = (0,1)$ on the circle and a second point $P_2 = (x,y)$ is selected at random in $\mathbb{R}^2$.
Determine the probability distribution for the length of the chords, and its expectation value.
Hint: What is the probability density to find the second point in a direction ϕ with respect to the x-axis, when looking from P_1?

(b) Pick as first point $P_1 = (0,1)$ on the circle, as before. However, now the second point $P_2 = (x,y)$ is selected at random in $[-L,L] \times [-L,L]$ for some $L \in \mathbb{R}^+$.
Determine the probability distribution for the length of the chords and its expectation value.
How does the result depend on L?

(c) Employ rotational symmetry and only consider horizontal lines (x, y_h) with $x \in \mathbb{R}$ and fixed y_h. Let y_h be uniformly distributed in $[-1, 1]$.
Determine the probability distribution for the length of the chords and its expectation value.

(d) Write a Python program to support your findings and explore other settings.