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Exercise 5 (hints for solution) *Bertrand's paradox*

A line is dropped randomly on a circle. The intersection will be a chord. What is the probability that the length of the chord is larger than that of a side of the inscribed equilateral triangle?

Bertrand's paradox states that the resulting probabiliy depends on the rule how the lines are selected. To understand this observation we consider intersections with the unit circle at the origin.

(a) Specify the line by two points: The first point is $P_1 = (0, 1)$ on the circle and a second point $P_2 = (x, y)$ is selected at random in $\mathbb{R}^2$.
Determine the probability distribution for the length of the chords, and its expectation value.
Hint: What is the probability density to find the second point in a direction ϕ with respect to the x-axis, when looking from P_1?

Let α be the angle between the line through $(0, 1)$ and the y-axis. Then the lenght of the chord is $2 \cos\alpha$, Moreover, the probability for α may be chosen to be flat for $0 < \alpha < \pi/2$, such that

$$P(\Delta) = P(\alpha)\,\frac{\mathrm{d}\alpha}{\mathrm{d}\Delta} = \frac{1}{2\pi}\left[1 - \frac{\Delta^2}{4}\right]^{-1/2}$$

such that

$$\langle\Delta\rangle = \int_0^2 \Delta\, P(\Delta) = \frac{4}{\pi}$$

(b) Pick as first point $P_1 = (0, 1)$ on the circle, as before. However, now the second point $P_2 = (x, y)$ is selected at random in $[-L, L] \times [-L, L]$ for some $L \in \mathbb{R}^+$.
Determine the probability distribution for the length of the chords and its expectation value.
How does the result depend on L?

(c) Employ rotational symmetry and only consider horizontal lines (x, y_h) with $x \in \mathbb{R}$ and fixed y_h. Let y_h be uniformly distributed in $[-1, 1]$.
Determine the probability distribution for the length of the chords and its expectation value.

In view of Pytagoras theorem the length of the chord is $2\sqrt{1 - y_h^2}$. By symmetry we may choose $y_h \in [0, 1]$ with $P(h) = 1$ such that

$$P(\Delta) = P(h)\left|\frac{\mathrm{d}h}{\mathrm{d}\Delta}\right| = \frac{\Delta}{4\sqrt{1 - \Delta^2/4}}$$

and expectation value

$$\langle\Delta\rangle = \int_0^2 \mathrm{d}\Delta\,\frac{\Delta^2}{4\sqrt{1 - \Delta^2/4}} = \frac{\pi}{2}$$

For comparison with the numerical data we will employ the cumulative distribution function

$$\mathcal{C}(\Delta) = \int_0^{\Delta} \mathrm{d}\tilde{\Delta}\, \frac{\tilde{\Delta}^2}{4\sqrt{1-\tilde{\Delta}^2/4}} = 1 - \sqrt{1-\frac{\Delta^2}{4}}$$

(d) Write a Python program to support your findings and explore other settings.

Install `Jupyter` as described on the page `https://jupyter.org/install`.

In Linux Jupyter can be started from the command line by typing `jupyter notebook`. In your browser you can then open the document `blatt02__bertrands_paradox.ipynb`.

Check the results for the following values of input parameters

samples	1000	10000	100000	100000	100000	100000
Lrange	30	30	30	10	3	1

What do you observe? Do you understand the parameter dependence?
Adapt the code to discuss the solution of part (b).