## Stochastic Processes

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Tutorial 1
Probability Measures
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## Exercise $1 \quad \sigma$ algebras for throwing dice

We consider the outcomes of rolling a die with faces $\Omega=\{1,2,3,4,5,6\}$. The die is a cube with three independent axes that correspond to the sides $X=\{1,6\}, Y=\{2,5\}$, and $Z=\{3,4\}$, respectively.
(a) Construct the $\sigma$-algebra that admits distinction only of the axes.
(b) Provide a generating set for this $\sigma$ algebra.
(c) We throw two dice and are interested in the overall sum of points. What are the mutually exclusive sets of events that can be used to generate the algebra for this problem? What are probabilities of the events?
(d) We again throw two dice, but we are only interested in doubles, i.e., outcomes where both dice show the same number of points. How is this problem related to Russion roulette?

## Exercise 2 Probabiliy of non-exclusive events

The events $X, Y$ and $Z$ in exercise 1 are mutually exclusive, and for a fair die they appear with equal probability.
(a) What is the probability for the events $A=\{X, Y\}$ that either $X$ or $Y$ is encountered, and for $B=\{X, Z\}$ that either $X$ or $Z$ is encountered. Provide an intuitive and a formal argument.
(b) What are the probabilities for $\Pi(A \cup B)$ and $\Pi(A \cap B)$ ?
(c) Use the axioms for probabilities to derive a general relation between the probabilities $\Pi\left(M_{1}\right)$, $\Pi\left(M_{2}\right), \Pi\left(M_{1} \cup M_{2}\right)$, and $\Pi\left(M_{1} \cap M_{2}\right)$.

## Exercise 3 Random walks (discrete in space and time)

We gamble on head and tail for coin flipping and certain sets of outcomes of dice. Proof the following statements.
(a) When the coin tossing is fair, the probability to encounter $h$ heads in $N$ tosses is

$$
\begin{equation*}
P(h, N)=\frac{N!}{h!(N-h)!2^{N}} \tag{1a}
\end{equation*}
$$

When it is head we win one Euro, for tail we loose one Euro. How much money do we expect to win after $N$ tosses, and what is the variance?
(b) When heads are encountered with probability $p$ in every single flip, the probability to encounter $h$ heads in $N$ tosses is

$$
\begin{equation*}
P(h, N)=\binom{N}{h} p^{h}(1-p)^{N-h} \tag{1b}
\end{equation*}
$$

How much money do we expect to win after $N$ tosses, and what is the variance?
(c) Now we play with a loaded die where we distinguish the events $\Omega=\{X, Y, Z\}$ - as defined in exercise 1. They correspond to winning $w_{i}, i \in \Omega$ Euro (negative values imply that one looses). Show that the probabily to encounter $n_{i}$ encounters of event $i \in \Omega$ in $N=\sum_{i \in \Omega} n_{i}$ throws of the die is

$$
\begin{equation*}
P\left(n_{X}, n_{Y}, N\right)=\binom{N}{n_{X} n_{Y} n_{Z}} p_{X}^{n_{X}} p_{Y}^{n_{Y}} p_{Z}^{n_{Z}} \tag{1c}
\end{equation*}
$$

where $p_{i}$ denotes the probability to enounter event $i \in \Omega$.
What is the expected gain after $N$ throws?
(d) When $w_{Y}=w_{Z}$ it is sufficient to distinguish the events $\{X\}$ and $\{Y, Z\}$.

Show that the the expected gain after $N$ throws can then be obtained from Eq. (1b) with a substantially simpler calculation.
In order to understand why this must be the case: Recover distribution Eq. (1b) by summation of Eq. (1c) over all admissible values of $n_{Y}$ and subsequently identifying $n_{X}$ with $h$. What is the relation between $p$ and $\left\{p_{X}, p_{Y}, p_{Z}\right\}$ ?

## Exercise 4 Generating function for random walks

The calculations of the moments in exercise 3 simplify a lot when one considers generating function for the gain. Let us gain the amount $x$ and $y$ upon encountering head or tail in coin tossing. Hence, after encountering $h$ heads in $N$ tosses we gain $g_{N}=x h+y(N-h)$. Therefore, the $n^{\text {th }}$ moment of the gain distribution amounts to

$$
\begin{align*}
\left\langle g_{N}^{n}\right\rangle & =\left\langle(x h+y(N-h))^{n}\right\rangle \\
& =\sum_{h=0}^{N}(x h+y(N-h))^{n}\binom{N}{h} p^{h}(1-p)^{N-h} \\
& =\frac{\mathrm{d}^{n}}{\mathrm{~d} t^{n}} \sum_{h=0}^{N} \mathrm{e}^{(x h+y(N-h)) t}\binom{N}{h} p^{h}(1-p)^{N-h}=\frac{\mathrm{d}^{n}}{\mathrm{~d} t^{n}} G_{N}(t) \\
\text { with } \quad G_{N}(t) & :=\left\langle\mathrm{e}^{g_{N} t}\right\rangle=\left(p \mathrm{e}^{x t}+(1-p) \mathrm{e}^{y t}\right)^{N} \tag{2}
\end{align*}
$$

(a) Use Eq. (2) to verify that the binomial distribution, Eq. (1b), is normalized and to calculate the first three moments of the expected gain.
(b) Determine a generating function for the gain in exercise 3(c), and calculate the expectation and the variance of the expected gain.

## Exercise 5 The characteristic function of the Lorentz-Cauchy distribution

The Lorentz-Cauchy distribution is defined as

$$
\begin{equation*}
p_{L C}(x)=\frac{1}{\pi} \frac{\Gamma}{(x-m)^{2}+\Gamma^{2}} \quad \text { with parameters } m, \Gamma \in \mathbb{R} \tag{3a}
\end{equation*}
$$

(a) Verify that the distribution is normalized.
(b) Determine the expectation value of the distribution.

What about the variance?
(c) Show that the characteristic function of $p_{L C}(x)$ is

$$
\begin{equation*}
\chi_{L C}(t)=\left\langle\mathrm{e}^{\mathrm{i} t x}\right\rangle=\mathrm{e}^{\mathrm{i} m t-\Gamma|t|} \tag{3b}
\end{equation*}
$$

What does this tell about the normalization, expectation and variance?
(bonus) Consider the distribution

$$
\begin{equation*}
p_{4}(x)=\frac{4}{\pi} \frac{\Gamma^{3}}{(x-m)^{4}+4 \Gamma^{4}} \quad \text { with parameters } m, \Gamma \in \mathbb{R} \tag{3c}
\end{equation*}
$$

Determine the characteristic function.
Show that it provides $m$ and $2 \Gamma^{2}$ for the expectation and the variance, respectively. What happens for higher moments?

