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Quantum Field Theory — Problem Sheet 5

2 pages — Problems 5.1 to 5.4

Problem 5.1
Let $\mathfrak{A}$ be a unital C^* subalgebra of $\mathscr{B}(\mathcal{H})$. Then $\mathfrak{A}$ is called *irreducibly represented* on $\mathcal{H}$ if the $B \in \mathscr{B}(\mathcal{H})$ which commute with all $A \in \mathfrak{A}$ are exactly the multiples of the unit operator. One can write this condition equivalently as $\mathfrak{A}' = \mathbb{C}\mathbf{1}$ where $\mathfrak{A}'$ denotes the commutant of $\mathfrak{A}$. Show that the following statements are equivalent.

(i) $\mathfrak{A}$ is irreducibly represented on $\mathcal{H}$.

(ii) $\mathfrak{A}'' = \mathscr{B}(\mathcal{H})$.

(iii) All non-zero vectors in $\mathcal{H}$ are cyclic for $\mathfrak{A}$, i.e. if $\psi \in \mathcal{H}$, $\psi \neq 0$, then

$$\mathfrak{A}\psi = \{A\psi : A \in \mathfrak{A}\}$$

is a dense subspace of $\mathcal{H}$.

Problem 5.2
Show that the Wightman functions $\mathtt{W}_n(\xi_1, \ldots, \xi_{n-1})$ in the relative position variables (cf. Prob. 4.1) are distributional boundary values of analytic functions
$F_n(\xi_1 + iq_1, \ldots, \xi_{n-1} + iq_{n-1})$ for $\xi_j \in \mathbb{R}^4$, $q_j \in J^-(0)$, in the limit as $q_j \to 0$ for all j.

Problem 5.3
Let $\Phi, \mathcal{H}, \mathcal{D}, \mathscr{U}, \Omega$ be a Wightman quantum field. Let $u : \mathscr{S}(\mathbb{R}^4) \to \mathbb{C}$ be a distribution, and define a new quantum field $\Phi_u(f) = \Phi(f) + u(f)\mathbf{1}$. Show that

$$\mathcal{A}(O) = \mathcal{A}_u(O) \quad \text{for any double cone } O\,,$$

where $\mathcal{A}(O)$ is the local operator algebra generated by polynomials in the $\Phi(f)$ and $\mathcal{A}_u(O)$ is the local operator algebra generated by polynomials in the $\Phi_u(f)$, with $\mathrm{supp}(f) \subset O$.

Problem 5.4
Let $H = H_0 + H_I$ be a Hamiltonian defined on a dense and invariant domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$. The Hamiltonian is split in an "interaction free part" H_0 and some "interaction part" H_I which is assumed to be a "small perturbation". Then one can define, for any unit vector $\psi \in \mathcal{D}$, asymptotic states $\psi_\pm$ defined by

$$\lim_{\pm t \to \infty} ||\mathrm{e}^{-itH}\psi_\pm - \mathrm{e}^{-itH_0}\psi|| = 0$$

provided the limit exists. This then defines the wave-operators

$$W_{\pm} = \lim_{\pm t \to \infty} \mathrm{e}^{-itH} \mathrm{e}^{-itH_0}$$

from which the S-matrix $S = W_+^* W_-$ is constructed, provided the $W_{\pm}$ turn out to be unitary operators.

(a) Discuss, at a conceptual level, the similarity (and potential differences) to the construction of asymptotic scattering states in Haag-Ruelle scattering theory. In particular, identify corresponding roles of the interacting/interaction-free theories.

(b) Define $H_{\text{int}}(t) = \mathrm{e}^{itH_0} H_I \mathrm{e}^{-itH_0}$, the interaction part in the interaction picture. Show that

$$S = \sum_{n=0}^{\infty} S_n \quad \text{where } S_0 = \mathbf{1}\,, \;\; S_n = \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \, T(H_{\text{int}}(t_1) \cdots H_{\text{int}}(t_n)) \;\; (n \geq 1)$$

Here, $T(\cdots)$ means time-ordering, i.e. the operators in the product are ordered from the left according to decreasing times. Assume that everything converges as required. This is also formally written as "time-ordered exponential"

$$S = T\text{-}\exp(-i \int dt \, H_{\text{int}}(t))$$